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Arens Multiplications on Locally Convex Completions of Banach Algebras.

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ARENS MULTIPLICATIONS ON LOCALLY CONVEX
COMPLETIONS OF BANACH ALGEBRAS

A Dissertation

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The Department of Mathematics

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Edith Anne McCharen
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TABLE OF DEFINITIONS AND SYMBOLS

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$E', (E, \mathfrak{J})^*, (E, \mathfrak{J})^{*'} \dots\dots\dots$	4
$\sigma(E, F) \dots\dots\dots$	5
$\tau(E, F) \dots\dots\dots$	6
polar of $U \subset E$ taken in $E', U^\circ = \{f \in E' : f(u) \leq 1 \forall u \in U\} \dots\dots$	6
$B(H)$, WOT, weak operator topology $\dots\dots\dots$	11
$f \cdot a, a \cdot f \dots\dots\dots$	13
$\beta, \beta^1, \beta^2 \dots\dots\dots$	16
$\mu, \mu^1, \mu^2 \dots\dots\dots$	17
$\mathscr{A}, \mathscr{A}^1, \mathscr{A}^2, P \dots\dots\dots$	18
$F \cdot f, F:f \dots\dots\dots$	31
Arens multiplications, regularity, $FG, F:G \dots\dots\dots$	32
$\mu_e, \mu_e^1, \mu_e^2 \dots\dots\dots$	36
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$\mathscr{A}_e \dots\dots\dots$	45
$(E, \mathfrak{J})^\wedge \dots\dots\dots$	76
$L(A), R(A), M(A) \dots\dots\dots$	84
$\mathcal{L}(A), \mathcal{R}(A), \mathcal{A}(A) \dots\dots\dots$	92
$C(A)$, the B^* -algebra enveloping A , $ a , P' \dots\dots\dots$	118
SOT, strong operator topology $\dots\dots\dots$	147

ABSTRACT

In this paper we investigate whether certain locally convex completions of a Banach algebra are algebras under multiplications defined analogously to those which R. Arens defined on the bidual of a Banach algebra.

The first chapter establishes notation and terminology. In Chapter II we define three families of topologies -- the β , β' , and μ families -- which are under consideration in the sequel. We determine their topological duals and discuss some of the basic relationships among these topologies. The first family is the left, right, and two-sided strict topologies on a Banach algebra with bounded approximate identity; the second consists of the analogues of Sakai's strong topology and Akemann's strong* topologies on W^* -algebras in the setting of a Banach *-algebra; and the third family is a set of three new topologies defined on a Banach algebra with bounded approximate identity.

Chapter III is concerned with the Arens multiplications on the bidual of a given Banach algebra A . In the first section we extend the μ topologies to the bidual of A and show that their relationships to $\tau(A^{**}, A^*)$ are related to the question of regularity of the Arens multiplications on A^{**} . In the second section we define an AB^* -algebra (of

which a B^* -algebra is an example) and show that the Arens multiplications are always regular on the biduals of B^* and AB^* -algebras with bounded approximate identity. Moreover, we give an extension of one of the \mathscr{A} topologies to the bidual of a Banach algebra with bounded approximate identity; its relationship to $\sigma(A^{**}, A^*)$ distinguishes B^* , AB^* , and non- AB^* algebras. Tomita's proof that the bidual of a B^* -algebra is also B^* provides the motivation for this section and is included. The third section contains various examples, and a few basic properties of AB^* -algebras are also developed here.

Chapter IV is concerned with the completions of a Banach algebra with respect to the preceding topologies. In the first section we give necessary conditions that an arbitrary locally convex completion be an algebra under either Arens multiplication. The remaining three sections are concerned specifically with showing that the β , \mathscr{A} , and μ completions of certain Banach algebras are algebras under at least one Arens multiplication provided that certain conditions are satisfied.

The topologies \mathscr{A} and μ in the second and third families have analogues in the setting of W^* -algebras, and we discuss them briefly in Chapter V.

In conclusion Chapter VI contains several interesting questions that have arisen in connection with this paper.

INTRODUCTION

R. Arens [2] has defined two multiplications on the bidual of a Banach algebra A so that the bidual A^{**} is an algebra: one multiplication extends the given multiplication on A and the other extends its transpose. We considered the following problem: if \mathfrak{T} is a locally convex Hausdorff topology on A , what conditions on \mathfrak{T} imply that multiplications analogous to those defined by Arens can be defined on the \mathfrak{T} -completion of A and that it be an algebra with respect to these multiplications.

We have sufficient conditions that this be the case; however, the conditions seem difficult to verify without knowing a great deal about \mathfrak{T} and the structure of A . Therefore, we examined the problem from the standpoint of examples.

We define three families of topologies, each family consisting of three topologies. The first family is the left, right, and two-sided strict topologies on a Banach algebra with bounded approximate identity; the second consists of the analogues of Sakai's strong topology [20] and Akemann's strong* topology [1] on W^* -algebras in the setting of a Banach $*$ -algebra; and the third family is a set of three new topologies defined on a Banach algebra with

bounded approximate identity. It was recently brought to our attention that P. C. Shields [24] first introduced on W^* -algebras a topology analogous to one of the topologies in our third family; however, our discovery and investigation of this family of topologies was made independently.

Consideration of the second family of topologies leads us to define a new type of Banach algebra -- an AB^* -algebra -- which includes B^* -algebras as a special case. We give two examples of AB^* -algebras which are not B^* under any equivalent norm, and we establish some of the basic properties of AB^* -algebras.

Using entirely different techniques for each of the families of topologies, we show that the completions with respect to two of the three topologies in each family are algebras under some Arens multiplication. The remaining topology in each family presents a problem: we can prove that their completions are algebras under some Arens multiplication only with additional assumptions, some of which are quite restrictive.

To establish some of the foregoing results, we extend certain topologies to the biduals of the Banach algebras on which they are defined and examine some of their properties. It happens that the relationships of these extensions to $\sigma(A^{**}, A^*)$ and $\tau(A^{**}, A^*)$ are interconnected with algebraic properties of the Arens multiplications on A^{**} : namely,

whether multiplicative identities exist and whether the Arens multiplications on A^{**} are transposes of one another. The case of the bidual of a B^* -algebra is especially interesting: we include Tomita's proof that it is actually a W^* -algebra [27], and we prove that A is a two-sided ideal in A^{**} if and only if A is dual.

The topologies in the second and third families also have analogues in the setting of W^* -algebras, and we discuss this briefly. In particular, we consider Akemann's criterion for a subset of the predual of a W^* -algebra to be weakly relatively compact, which is formulated in terms of the \mathscr{L}_e seminorms on a W^* -algebra. We show that for a certain type of Banach algebra A a subset of A^* satisfies Akemann's criterion if and only if it is equicontinuous with respect to a topology on A related to the \mathscr{L} topology, and we apply this result to the case where A is an AB^* -algebra. We also give a counterexample to a claim by P. C. Shields in which he asserts that multiplication on a W^* -algebra is jointly μ_e -continuous provided that one variable remains bounded. In addition to these topics we extend some of the previous results on relationships among the topologies in the three families.

In conclusion we list several interesting questions that have arisen in connection with this paper.

To facilitate reading, the numbering of definitions,

lemmas, and theorems is consecutive within a section and is done in triple digits standing for chapter, section, and paragraph, respectively. A table of definitions and symbols is also included.

CHAPTER I

PRELIMINARIES

The purpose of this chapter is to establish notation and terminology, and to state some of the more frequently used results about locally convex spaces and Banach algebras. Our standard references are Robertson and Robertson [18] and Rickart [16]. In particular, for B^* -algebras we use Dixmier [7].

Section 1. Locally Convex Spaces.

Let E be a vector space over the complex field. The linear span of a subset X in E is denoted by $\langle X \rangle$. If X and Y are subsets of E , the set of elements of the form $x+y$, where x is in X and y is in Y , is denoted by $X+Y$.

If E and F are vector spaces and $f:E \rightarrow F$ is a linear mapping, we say that f is an injection if its kernel is the zero vector of E , and that f is a surjection if the set $f^{-1}(y)$ is non-empty for every y in F .

Let \mathcal{T} be a topology on E for which E is a topological vector space. A set U is called a neighborhood of the point x in E if there exists an open set V containing x which is contained in U . If the set U is a neighborhood of the zero vector in E , we simply call U a neighborhood.

The topology \mathfrak{T} is said to be locally convex if it has a base of absolutely convex neighborhoods. In this case, the topological vector space (E, \mathfrak{T}) is said to be a locally convex space.

Let \mathfrak{T} and \mathfrak{T}' be locally convex topologies on E . If every neighborhood in the \mathfrak{T} topology is a neighborhood in the \mathfrak{T}' topology, we say that \mathfrak{T}' is finer than \mathfrak{T} , (or, equivalently, that \mathfrak{T} is coarser than \mathfrak{T}'), and write this as $\mathfrak{T} \leq \mathfrak{T}'$; if $\mathfrak{T} \leq \mathfrak{T}'$ and $\mathfrak{T}' \leq \mathfrak{T}$, we say that the topologies \mathfrak{T} and \mathfrak{T}' are equivalent. The locally convex topology on E whose neighborhoods consist of all sets containing a set of the form $U \cap V$, where U is a \mathfrak{T} -neighborhood and V is a \mathfrak{T}' -neighborhood, is denoted by $\mathfrak{T} \vee \mathfrak{T}'$, and we note that $\mathfrak{T} \leq \mathfrak{T} \vee \mathfrak{T}'$ and $\mathfrak{T}' \leq \mathfrak{T} \vee \mathfrak{T}'$.

If (E, \mathfrak{T}) and (F, \mathfrak{T}') are convex spaces and $f: (E, \mathfrak{T}) \rightarrow (F, \mathfrak{T}')$ is a continuous mapping, we say that f is \mathfrak{T} - \mathfrak{T}' continuous. In case $E = F$ and $\mathfrak{T} = \mathfrak{T}'$, we simply say that f is \mathfrak{T} -continuous. If E and F are normed spaces, a mapping f that is continuous with respect to the norm topology on each space is called a norm-continuous mapping, or simply a continuous mapping.

The theorems stated in this section without proof can be found in [18].

1.1.1 Theorem. Let E be a vector space over the

complex field. If p and q are two seminorms on E such that $q(x) \leq 1$ whenever $p(x) \leq 1$, then $q(x) \leq p(x)$ for all x in E .

1.1.2 Theorem. Given any (non-empty) set Q of seminorms on a vector space E , there is a coarsest topology on E for which E is a topological vector space and in which every seminorm of Q is continuous. This topology is locally convex, and a base of closed absolutely convex neighborhoods is formed by the sets $\{x \in E : p_i(x) \leq \epsilon, i = 1, 2, \dots, n\}$, where p_1, \dots, p_n is any finite set of elements in Q and $\epsilon > 0$.

We call this coarsest topology the topology generated by Q . We have as a corollary to Theorem 1.1.2 that the topology generated by a set of seminorms Q on E is Hausdorff if and only if for each non-zero element x in E there is some p in Q with $p(x) \neq 0$.

1.1.3 (Hahn-Banach) Theorem. If p is a seminorm on the vector space E , and if f is a linear functional defined on a subspace M of E with $|f(x)| \leq p(x)$ for all x in M , then there is a linear functional \bar{f} defined on E which extends f and has the property that $|\bar{f}(x)| \leq p(x)$ for all x in E .

The vector space of all linear functionals on a vector

space E is called the algebraic dual of E and is denoted by E' . Let (E, \mathfrak{J}) be a locally convex space. Then the subspace of E' consisting of those linear functionals which are continuous in the \mathfrak{J} topology is called the topological dual of E with respect to \mathfrak{J} and is denoted by $(E, \mathfrak{J})^*$. If E is a normed space, we denote its topological dual with respect to the norm topology by E^* . The algebraic dual of E^* is denoted by $E^{*'} ;$ and since E^* is a normed space in a natural way, the topological dual of E^* with respect to this norm is denoted by E^{**} and is called the bidual of E .

For any normed space E there is a unique Banach space E^\wedge and an injection $\tau: E \rightarrow E^\wedge$ satisfying the following properties: (i) τ is linear; (ii) τE is dense in E^\wedge and (iii) for every e in E , the norm of τe in E^\wedge is equal to the norm of e in E . If $e \mapsto \|e\|$ denotes the norm in E , the Banach space E^\wedge is called the completion of E with respect to the norm $e \mapsto \|e\|$.

1.1.4 (Open Mapping) Theorem. Let E and F be Banach spaces and $f: E \rightarrow F$ be a continuous linear surjection. Then if f is injective, the mapping f^{-1} is continuous.

Let E and F be vector spaces and $E \times F$ be their Cartesian product. Suppose there is a complex-valued function defined on $E \times F$, whose value at a pair (x, y) will

be denoted by $\langle x, y \rangle$, which satisfies the following two conditions: (i) the function $\langle x, y \rangle$ is linear in x for fixed y and linear in y for fixed x ; and (ii) for each non-zero element y in F there is an element x in E with $\langle x, y \rangle \neq 0$, and for each non-zero element x in E there is an element y in F with $\langle x, y \rangle \neq 0$. Then the vector spaces E and F are said to be a dual pair. For instance, if (E, \mathcal{J}) is a locally convex Hausdorff space, then E and $(E, \mathcal{J})^*$ are a dual pair under the function $\langle x, y \rangle = f(x)$ by the Hahn-Banach theorem.

Let E and F be a dual pair and X be a subset of E . Then X is said to separate points of F if for every non-zero element y in F there is an element x in X such that $\langle x, y \rangle \neq 0$.

If E and F are a dual pair, the locally convex Hausdorff topology on E generated by the seminorms $\{e \mapsto |\langle e, f \rangle| : f \in F\}$ is called the weak topology on E determined by F and is denoted by $\sigma(E, F)$. If G is a subspace of F , it is convenient to consider the topology on E generated by the seminorms $\{e \mapsto |\langle e, g \rangle| : g \in G\}$ even if E and G are not a dual pair; this topology, denoted by $\sigma(E, G)$, need not be Hausdorff.

If E and F are a dual pair, the topology of uniform convergence on the set of all absolutely convex $\sigma(F, E)$ -compact subsets of F is a locally convex Hausdorff

topology on E ; it is called the Mackey topology on E determined by F and is denoted by $\tau(E, F)$.

Let E and F be a dual pair. For each e in E let \bar{e} be the element of F' defined by $\bar{e}(f) = \langle e, f \rangle$. Similarly, for each f in F let \bar{f} be the element of E' defined by $\bar{f}(e) = \langle e, f \rangle$.

1.1.5 (Mackey-Arens) Theorem. Let E and F be a dual pair, and let \mathfrak{J} be a locally convex Hausdorff topology on E such that (E, \mathfrak{J}) is a topological vector space. Then $(E, \mathfrak{J})^* = \{\bar{f} : f \in F\}$ if and only if $\sigma(E, F) \leq \mathfrak{J} \leq \tau(E, F)$.

1.1.6 Theorem. Let E and F be a dual pair and G be a subspace of F' containing the set $\{\bar{e} : e \in E\}$. Then the bipolar $X^{\circ\circ}$ in G of a subset X of E is the $\sigma(G, F)$ -closed convex hull of X .

1.1.7 Theorem. If (E, \mathfrak{J}) is a locally convex Hausdorff space and U is a \mathfrak{J} -neighborhood in E , then U° in E' is $\sigma(E', E)$ -compact. In fact, since U° is actually contained in $(E, \mathfrak{J})^*$, the set U° is certainly $\sigma((E, \mathfrak{J})^*, E)$ -compact.

1.1.8 (Goldstine) Theorem. Let A be a Banach space. Then the image in A^{**} of the unit ball of A under the natural imbedding is $\sigma(A^{**}, A^*)$ -dense in the unit ball of A^{**} .

Proof. Let S denote the unit ball of A^{**} and let

us identify S with its image in A^{**} under the natural imbedding. Then $S^{\circ\circ}$ in A^{**} is the $\sigma\text{--}(A^{**}, A^*)$ -closed convex hull of S by Theorem 1.1.6. Moreover, $S^{\circ\circ}$ is the unit ball of A^{**} . The conclusion of the theorem now follows.

1.1.9 Corollary. If F is an element of A^{**} , there exists a net $\{a_\alpha\}$ in A such that $\|a_\alpha\| \leq \|F\|$ for each α , and F is the $\tau(A^{**}, A^*)$ -limit of the net $\{a_\alpha\}$.

Proof. Let S denote the unit ball of A imbedded in A^{**} . Then the $\tau(A^{**}, A^*)$ -closure of S equals the $\sigma(A^{**}, A^*)$ -closure of S by [18, p. 34] since S is convex. The proof is easily finished by applying Theorem 1.1.8.

Let E and F be Banach spaces and $f: E \rightarrow F$ be a linear mapping. The mapping f is said to be weakly compact if the image of the unit ball of E under the mapping f is a relatively $\sigma(F, F^*)$ -compact subset of F .

1.1.10 (Generalized Gantmacher) Theorem. Let E and F be Banach spaces and $f: E \rightarrow F$ be a continuous linear mapping. Then the following are equivalent: (i) the mapping f is weakly compact; (ii) the mapping $f^*: F^* \rightarrow E^*$, defined by $f^*(y)(e) = y(f(e))$ for y in F^* and e in E , is weakly compact; and (iii) the range of the mapping $f^{**}: E^{**} \rightarrow F^{**}$, defined by $f^{**}(x)(y) = x(f^*(y))$ for x in E^{**} and y

in F^* , is contained in F (identified with its natural image in F^{**}).

Proof. The reader is referred to [8, pp. 624-5].

Section 2. Banach Algebras.

Some of the definitions and results in this section are barely touched upon: they are not used in the main part of the paper but only in examples.

If A is a (complex) Banach algebra, let A_1 denote the (vector space) direct sum of A with the complex field. Then the elements of A_1 are pairs of the form (x, λ) , where x is in A and λ is a complex number. Defining multiplication by the relation $(x, \lambda)(x', \lambda') = (xx' + \lambda x' + \lambda' x, \lambda \lambda')$, and a norm by the relation $\|(x, \lambda)\| = \|x\| + |\lambda|$, we see that A_1 becomes a Banach algebra with identity element $(0, 1)$. If we set $(0, 1) = e$ and identify each x in A with $(x, 0)$ in A_1 , we can realize A_1 as the set of elements of the form $x + \lambda e$, with x in A and λ a complex number; moreover, the operations on A_1 become similar to those on a polynomial ring. The algebra A_1 is called the algebra formed from A by adjunction of an identity.

Let A be a Banach algebra. A net $\{e_\alpha : \alpha \in I\}$ in A , (where I is a directed set), is called a bounded approximate identity for A if the following properties are satisfied: (i) $\|e_\alpha\| \leq 1$ for every α in I ; and (ii)

$\lim \|xe_\alpha - x\| = 0 = \lim \|e_\alpha x - x\|$ for every x in A . If a net $\{e_\alpha : \alpha \in I\}$ in A has only the properties that $\|e_\alpha\| \leq 1$ for every α in I , and that $\lim \|e_\alpha x - x\| = 0$ for every x in A ; then the net $\{e_\alpha\}$ is said to be a left bounded approximate identity; a right bounded approximate identity is defined analogously.

An element x of a (complex) algebra A is said to be quasi-regular if there is an element y in A such that $x + y - xy = 0 = x + y - yx$; an element of A is called quasi-singular if it is not quasi-regular. The spectrum of an element x in A , denoted by $Sp_A(x)$, is defined as follows: (i) if A does not have an identity, $Sp_A(x)$ is the set of all complex numbers λ such that $\lambda^{-1}x$ is quasi-singular and the number 0; (ii) if A has an identity, $Sp_A(x)$ is the set of all complex numbers λ such that $\lambda^{-1}x$ is quasi-singular, and the number 0 only in the case that x does not have a multiplicative inverse in A .

Let A be a commutative (complex) algebra, and denote by Φ the set of all non-zero homomorphisms of A into the complex field. If Φ is non-empty, there is associated to each element x of A a complex-valued function \hat{x} on Φ defined by $\hat{x}(\varphi) = \varphi(x)$. The A -topology is a topology defined on Φ by taking neighborhoods of the points φ in Φ to be all sets of the form $\{\varphi' : |\hat{x}_i(\varphi') - \hat{x}_i(\varphi)| < \epsilon, i=1,2,\dots,n\}$, where x_1, \dots, x_n is any finite set of elements in A and

$\epsilon > 0$. The set Φ endowed with the A -topology is called the carrier space of the algebra A . The mapping $x \rightarrow \hat{x}$, called the Gelfand transform on A , maps A into $C(\Phi)$, the set of complex-valued continuous functions on Φ .

A Banach *-algebra is a Banach algebra with an involution satisfying the property that $\|x^*\| = \|x\|$ for every element x in the algebra. A Banach *-algebra is said to be symmetric if every element of the form $-x^*x$ is quasi-regular.

An element x of a Banach *-algebra A is said to be hermetian if $x = x^*$. In particular, if A has an identity, it is an hermetian element. If A has a bounded approximate identity $\{e_\alpha\}$, then it has an hermetian bounded approximate identity:—for example, take the net $\{\frac{1}{2}(e_\alpha + e_\alpha^*)\}$. Involution on A is said to be hermetian if $\text{Sp}_A(x)$ is real for every hermetian element x in A .

Let A be a Banach *-algebra. A linear functional ψ on A is said to be positive if $\theta(x^*x) \geq 0$ for every x in A .

1.2.1 Cauchy-Schwartz Inequality. If θ is a positive linear functional on a Banach *-algebra A , then $|\theta(xy)|^2 \leq \theta(xx^*) \theta(y^*y)$ for every x and y in A .

Proof. The reader is referred to [16, p. 213].

1.2.2 Theorem. If A is a Banach $*$ -algebra with bounded approximate identity, any positive linear functional θ on A is continuous. Moreover $\overline{\theta(x)} = \theta(x^*)$ for every x in A .

Proof. For the continuity of positive linear functionals the reader is referred to [28]; the other assertion follows from this and [7 ,p. 23].

A Banach $*$ -algebra A is said to be a B*-algebra if $\|x\|^2 = \|x^*x\|$ for every x in A , and this relation on the norm is called the B*-condition. A Banach $*$ -algebra is said to be a B*-algebra under an equivalent norm if there exists a norm on the algebra satisfying the B*-condition which is equivalent to the given norm.

1.2.3 Theorem. Any B*-algebra A has a bounded approximate identity, and its topological dual is spanned by the continuous positive linear functionals on A .

Proof. The reader is referred to [7 ,pp.15 and 40].

If H is a Hilbert space, we indicate the inner product of x and y in H by $((x,y))$. Let $B(H)$ denote the set of bounded linear operators on H ; we assume that the reader is familiar with its basic properties. The Hausdorff topology generated on $B(H)$ by the seminorms $\{S \mapsto |((Sx,y))| : x,y \in H\}$ is called the weak operator topology on $B(H)$ and is denoted by WOT. Other topologies on $B(H)$

are discussed in Chapter V.

Let A be a Banach algebra and $x \mapsto x^*$ be a mapping on A satisfying the following properties for every x and y in A and every complex number λ : (i) $(x^*)^* = x$; (ii) $(x + y)^* = x^* + y^*$; and (iii) $(\lambda x)^* = \bar{\lambda}x^*$. Notice that the mapping $x \mapsto x^*$ on A satisfies all the properties of an involution with the exception that $(xy)^*$ need not be y^*x^* for x and y in A . Then a $*$ -representation of A is a mapping T from A into some $B(H)$, which satisfies the following properties: (i) T is an algebra homomorphism; and (ii) $T(a^*) = T(a)^*$ for every a in A , where $T(a)^*$ indicates the adjoint of the operator $T(a)$. A $*$ -representation $T: A \rightarrow B(H)$ is said to be cyclic if there is a vector h in H such that the set $\{T(a)h: a \in A\}$ is dense in H ; any such vector h is called a cyclic vector.

We assume no continuity conditions on a $*$ -representation unless otherwise stated; however, sometimes a $*$ -representation by virtue of its existence is norm-continuous. For instance, if A is a Banach $*$ -algebra with identity, any $*$ -representation of A is norm-continuous [15,p. 241].

Section 3. Factorization Theorems.

Let A be a Banach algebra. A Banach space V is said to be an algebraic left A -module if there exists a

mapping from $A \times V$ into V , whose value at a pair (a, v) will be denoted by $a \cdot v$, which satisfies the following properties: (i) $a \cdot v$ is linear in a for fixed v and linear in v for fixed a ; and (ii) $(ab) \cdot v = a \cdot (b \cdot v)$ for every a and b in A and v in V . The mapping $(a, v) \rightarrow a \cdot v$ is called a (left) modular action.

The Banach space V is said to be an isometric left A -module if it is an algebraic left A -module with the modular action satisfying the inequality $\|a \cdot v\| \leq \|a\| \|v\|$, for every a in A and v in V .

The definition of an isometric right A -module is analogous; in this case, however, we let $(a, v) \rightarrow v \cdot a$ denote the modular action.

If A is a Banach algebra, then A is itself an isometric left and an isometric right A -module with respect to its multiplication. Moreover, the topological dual A^* is an isometric left and an isometric right A -module under the following definitions for f in A^* and a in A :

(i) $a \cdot f(x) = f(xa)$ for all x in A ; and (ii) $f \cdot a(x) = f(ax)$ for all x in A . We note that for a and b in A and f in A^* that $(a \cdot f) \cdot b = a \cdot (f \cdot b)$, and therefore write $a \cdot f \cdot b$ unambiguously to mean either $(a \cdot f) \cdot b$ or $a \cdot (f \cdot b)$.

If V is an isometric left (resp., right) A -module, let $A \cdot V$ (resp., $V \cdot A$) denote the set of elements in V

which can be written in the form $a \cdot v$ (resp., $v \cdot a$) for some a in A and some v in V .

1.3.1 (Rieffel) Theorem. Let A be a Banach algebra with bounded approximate identity $\{e_\alpha\}$, and let V be an isometric left (resp., right) A -module. Then the following are equivalent: (i) the linear span of $A \cdot V$ (resp., $V \cdot A$) is dense in V ; (ii) $\lim \|e_\alpha \cdot v - v\| = 0$ (resp., $\lim \|v \cdot e_\alpha - v\| = 0$) for every v in V ; and (iii) $V = A \cdot V$ (resp., $V \cdot A$).

Proof. The reader is referred to [17].

If A is a Banach algebra with bounded approximate identity, by Theorem 1.3.1, each element a in A can be written as $a = bc$, for some b and c in A .

1.3.2 Theorem. If A is a Banach $*$ -algebra with bounded approximate identity, for each element a in A there is an hermetian element b and an element c in A such that $a = bc$.

Proof. Let $\{e_\alpha\}$ be an hermetian bounded approximate identity for A . Examining Hewitt's proof that (ii) \Rightarrow (iii) in Theorem 1.3.1, we see that each a in A can be written $a = bc$, where b and c are in A and b is a (norm) limit of polynomials in $\{e_\alpha\}$ with positive coefficients. Since the element b is a limit of hermetian elements, it is

also hermetian.

1.3.3 Theorem. If A is a Banach algebra with bounded approximate identity, for any finite set of elements a_1, a_2, \dots, a_n in A there exist elements b, x_1, \dots, x_n in A such that $a_i = bx_i$ for $i=1, 2, \dots, n$. Moreover, if A is a Banach $*$ -algebra as well, the element b can be chosen hermetian.

Proof. For a proof of this and much more general results the reader is referred to [17], [5], and [22].

CHAPTER II

THE β , μ , AND \mathcal{J} FAMILIES OF TOPOLOGIES

Three families of topologies will be under consideration in the sequel. In the first section we define the β , μ , and \mathcal{J} families of topologies on a given Banach algebra A and characterize their topological duals as specific subsets of A^* . We discuss the relationships of the various topologies with one another in the second section.

Section 1. Definitions and their Topological Duals.

2.1.1. Definition of the β family. Let A be a Banach algebra with bounded approximate identity. For each a and b in A define $p_{a,b}(x) = \max\{\|ax\|, \|xb\|\}$ for x in A . Then each $p_{a,b}$ is a seminorm and the set $\{p_{a,b} : a, b \in A\}$ defines a locally convex topology on A called β . The topology β is Hausdorff: for $x \neq 0$ there exists an element e of the bounded approximate identity such that $\|x - xe\| \leq \frac{1}{2}\|x\|$; hence $\frac{1}{2}\|x\| \leq \|xe\| \leq p_{e,e}(x)$. The seminorms $x \mapsto \|ax\|$ (resp., $x \mapsto \|xa\|$) also define a locally convex Hausdorff topology called β^1 (resp., β^2).

Note that $\beta = \beta^1 \vee \beta^2$. A base for neighborhoods of zero for β^1 consists of sets of the form $\{x : \|ax\| \leq 1\}$. For

if a_1 and a_2 are in A , by [14] there exist c , d_1 , and d_2 in A such that $a_1 = d_1c$ and $a_2 = d_2c$. Therefore, let $e = (\|d_1\| + \|d_2\|)c$. Then $\max\{\|a_1x\|, \|a_2x\|\} \leq \max\{\|d_1\| \|cx\|, \|d_2\| \|cx\|\} \leq \|ex\|$.

If A has an involution, then sets of the form $\{x: \|ax\| \leq 1 \text{ and } \|xa\| \leq 1\}$ form a base for β . Let a and b be elements of A . Then $a = a_1 + ia_2$, where a_1 and a_2 are hermetian. It follows from Theorem 1.3.2 that there exist an hermetian element h of A and elements d_1, d_2 , and d_3 in A such that $a_1 = hd_1$, $a_2 = hd_2$, and $b = hd_3$. Let $e = (\|d_1^*\| + \|d_2^*\| + \|d_3\|)h$. Then $\max\{\|ax\|, \|xb\|\} \leq \max\{\|a_1x\| + \|a_2x\|, \|xb\|\} \leq \max\{\|d_1^*\| \|hx\| + \|d_2^*\| \|hx\|, \|xh\| \|d_3\|\} \leq \max\{\|ex\|, \|xe\|\}$.

2.1.2 Definition of the μ family. Let A be a Banach algebra with bounded approximate identity. For f in A^* define $p_f(x) = \max\{\|f \cdot x\|, \|x \cdot f\|\}$ for each x in A . Then each p_f is a seminorm and the set $\{p_f: f \in A^*\}$ defines a locally convex topology called μ . The topology μ is Hausdorff: for $x \neq 0$ there exists f in A^* such that $f(x) > 0$, and there exists an element e of the bounded approximate identity such that $\|x - xe\| \leq f(x)/2 \cdot \|f\|$; therefore $\frac{1}{2}f(x) \leq |f(xe)| \leq \|f \cdot x\| \leq p_f(x)$. The seminorms $x \mapsto \|f \cdot x\|$ (resp., $x \mapsto \|x \cdot f\|$) also define a locally convex Hausdorff topology called μ^1 (resp., μ^2).

A base for neighborhoods of zero for μ consists of sets of the form $\{x: \|f_i \cdot x\| \leq 1 \text{ and } \|x \cdot f_i\| \leq 1, i = 1, 2, \dots, n\}$, and $\mu = \mu^1 \vee \mu^2$.

2.1.3 Definition of the \mathcal{A} family. Let A be a Banach $*$ -algebra with bounded approximate identity, and let $P = \{\theta \in A^*: \theta(x^*x) \geq 0, \forall x \in A\}$. For θ in P define $p_\theta(x) = \max\{\theta(x^*x)^{\frac{1}{2}}, \theta(xx^*)^{\frac{1}{2}}\}$. Then each p_θ is a seminorm by the Cauchy-Schwartz inequality, and the set $\{p_\theta: \theta \in P\}$ defines a locally convex topology called \mathcal{A} . This topology is Hausdorff if and only if P separates points of A . Suppose for $x \neq 0$ there exists θ in P such that $|\theta(x)| > 0$; Then there exists e in the bounded approximate identity such that $\|x - xe\| \leq |\theta(x)|/2 \cdot \|\theta\|$, and so $\frac{1}{2}|\theta(x)| \leq |\theta(xe)| \leq \theta(xx^*)^{\frac{1}{2}} \theta(e^*e)^{\frac{1}{2}} \leq \|\theta\|^{\frac{1}{2}} \theta(xx^*)^{\frac{1}{2}} \leq \|\theta\|^{\frac{1}{2}} p_\theta(x)$. For the converse we show that $\theta(x) = 0$ for all θ in P implies that $\theta(x^*x) = 0$ and $\theta(xx^*) = 0$ for all θ in P . To that end, let $\{e_\alpha\}$ be an hermetian bounded approximate identity and λ be any complex number. Then for each θ in P the functional $(\lambda e_\alpha + x) \cdot \theta \cdot (\lambda e_\alpha + x)^*$ is in P , so by hypothesis $0 = (\lambda e_\alpha + x) \cdot \theta \cdot (\lambda e_\alpha + x)^*(x) = |\lambda|^2 \theta(e_\alpha x e_\alpha) + \lambda \theta(x^* x e_\alpha) + \bar{\lambda} \theta(e_\alpha x x) + \theta(x^* x x^*)$. Since $\|e_\alpha x e_\alpha - x\| \leq \|e_\alpha x e_\alpha - e_\alpha x\| + \|e_\alpha x - x\| \leq 2\|e_\alpha x - x\|$ and the last term of the inequality converges to 0, we have that $0 = |\lambda|^2 \theta(x) + \lambda \theta(x^*x) + \bar{\lambda} \theta(xx) + \theta(x^*xx^*) = \lambda \theta(x^*x) + \bar{\lambda} \theta(xx)$

$+ \theta(x^*xx^*)$. Letting $\lambda = 1, -1, i$, and $-i$; multiplying the resulting equations by $1, -1, -i$, and i respectively; and adding, we obtain $4\theta(x^*x) = 0$. Now $\theta(x^*) = \overline{\theta(x)} = 0$, so that by applying the same argument to x^* we have $\theta(xx^*) = 0$.

The seminorms $x \rightarrow \theta(x^*x)^{\frac{1}{2}}$ (resp., $x \rightarrow \theta(xx^*)^{\frac{1}{2}}$) also define a locally convex topology called \mathscr{J}^1 (resp., \mathscr{J}^2), which is Hausdorff if and only if \mathscr{J} is. Note that sets of the form $\{x: \theta(x^*x) \leq 1\}$ give a base for neighborhoods of zero for \mathscr{J}^1 , and that $\mathscr{J} = \mathscr{J}^1 \vee \mathscr{J}^2$.

We now examine some of the properties of these topologies.

2.1.4 Theorem. Let A be a Banach algebra with bounded approximate identity. Then (i) multiplication is separately continuous in each of the topologies of the β and μ families; (ii) we have the following orderings:

$\beta \leq \tau(A, A^*)$; $\sigma(A, A^*) \leq \mu^1 \leq \mu \leq \tau(A, A^*)$; and $\sigma(A, A^*) \leq \mu^2 \leq \mu \leq \tau(A, A^*)$. If A is a $*$ -algebra, then (iii) multiplication is separately continuous for each of the \mathscr{J} topologies; (iv) $\mathscr{J} \leq \tau(A, A^*)$; and (v) the mapping $a \rightarrow a^*$ is β , μ , and \mathscr{J} continuous.

Proof. Parts (i), (iv) and (v) are clear. In (ii), $\beta \leq \tau(A, A^*)$ since $\max\{\|ax\|, \|xb\|\} \leq (\|a\| + \|b\|)\|x\|$ for a, b , and x in A . The second part of (ii) follows from the

inequality $|f(x)| \leq \min\{\sup|f(xe_\alpha)|, \sup|f(e_\alpha x)|\}$
 $\leq \min\{\|f \cdot x\|, \|x \cdot f\|\} \leq \max\{\|f \cdot x\|, \|x \cdot f\|\} \leq \|f\| \|x\|$, for f in
 A^* , x in a , and $\{e_\alpha\}$ a bounded approximate identity for A .

Suppose now that A is a $*$ -algebra. Then for θ in P and a and b in A we have $a \cdot \theta \cdot a^*$ is in P and
 $|\theta(a^*ba)| \leq \|b\| \theta(a^*a)$ by [, p. 23]. Using these results,
 we see that for any \mathscr{J}^1 -neighborhood $W = \{x \in A; \theta(x^*x) \leq 1\}$ and
 any element a in A , the \mathscr{J}^1 -neighborhood
 $U = \{x \in A; a \cdot \theta \cdot a^*(x^*x) \leq 1\}$ has the property that $U \cdot a \subset W$,
 and the \mathscr{J}^1 -neighborhood $V = \{x \in A; \|a\|^2 \theta(x^*x) \leq 1\}$ has the
 property that $a \cdot V \subset W$. The proof that multiplication is
 separately continuous in the \mathscr{J}^2 topology is similar; the
 proof for the \mathscr{J} topology follows immediately from this since
 $\mathscr{J} = \mathscr{J}^1 \vee \mathscr{J}^2$.

From the Mackey-Arens Theorem we have the following:

2.1.5 Corollary. The topological duals of the topologies in
 the β , \mathscr{J} , and μ families, provided they are defined and
 Hausdorff on the Banach algebra A , are subsets of A^* ; and
 $(A, \mu)^* = (A, \mu^1)^* = (A, \mu^2)^* = A^*$.

We now compute the topological duals for the β
 family. The β^1 and β^2 duals are easily found by using
 [17]. D. C. Taylor [26] shows that if A is a B^* -algebra,
 then $(A, \beta)^* = \{f \in A^*: \lim \|e_\alpha \cdot f + f \cdot e_\alpha - e_\alpha \cdot f \cdot e_\alpha - f\| = 0\}$,
 where $\{e_\alpha\}$ is a bounded approximate identity for A . A

slight modification, making use of an auxiliary module over A [5] , of his proof removes this restriction on A .

2.1.6 Lemma. Let \mathfrak{J} and \mathfrak{J}' be locally convex Hausdorff topologies on a vector space E . Then $(E, \mathfrak{J} \vee \mathfrak{J}')^* = (E, \mathfrak{J}')^* + (E, \mathfrak{J})^*$ in E' .

Proof. Let \mathcal{U} be a base of neighborhoods for \mathfrak{J} consisting of absolutely convex closed neighborhoods such that for each U in \mathcal{U} the set λU is in \mathcal{U} for every non-zero complex number λ ; let \mathcal{V} be a base of neighborhoods for \mathfrak{J}' with the same properties.

Then $(E, \mathfrak{J})^* = \bigcup \{U^\circ : U \in \mathcal{U}\}$ and $(E, \mathfrak{J}')^* = \bigcup \{V^\circ : V \in \mathcal{V}\}$, where the polars are taken in E' [18, p. 35]. Moreover, the collection of sets of the form $U \cap V$, for U in \mathcal{U} and V in \mathcal{V} , form a base of neighborhoods for $\mathfrak{J} \vee \mathfrak{J}'$. Consequently, $(E, \mathfrak{J} \vee \mathfrak{J}')^* = \bigcup \{U \cap V\}^\circ; U \in \mathcal{U}, V \in \mathcal{V}$.

Each U in \mathcal{U} is convex and \mathfrak{J} -closed, and thus $\sigma(E, (E, \mathfrak{J})^*)$ -closed, similarly, each V in \mathcal{V} is $\sigma(E, (E, \mathfrak{J}')^*)$ -closed [18, p. 34]. Since $\sigma(E, (E, \mathfrak{J} \vee \mathfrak{J}')^*)$ is a finer topology than either $\sigma(E, (E, \mathfrak{J})^*)$ or $\sigma(E, (E, \mathfrak{J}')^*)$, U and V are closed in this finer topology. Since U and V are absolutely convex as well, the set $(U \cap V)^\circ$ is the $\sigma((E, \mathfrak{J} \vee \mathfrak{J}')^*, E)$ -closed absolutely convex hull of $U^\circ \cup V^\circ$ [18, p. 36]. Moreover, U° and V° are in $(E, \mathfrak{J} \vee \mathfrak{J}')^*$; since they are $\sigma(E', E)$ -compact [18, p. 62], they are clearly

$\sigma((E, \mathfrak{I} \vee \mathfrak{I}')^*, E)$ -compact. Hence the set $U^\circ + V^\circ$ is $\sigma((E, \mathfrak{I} \vee \mathfrak{I}')^*, E)$ -closed; it is absolutely convex and contains U° and V° . Thus, $(U \cap V)^\circ \subset U^\circ + V^\circ$.

Furthermore, $(2U)^\circ + (2V)^\circ \subset (U \cap V)^\circ$, for each U in \mathcal{U} and V in \mathcal{V} : choosing f in $(2U)^\circ$, g in $(2V)^\circ$, and x in $U \cap V$, we have that $2x$ is in $2U \cap 2V$ and that $2|(f+g)(x)| = |(f+g)(2x)| \leq |f(2x)| + |g(2x)| \leq 1 + 1$.

Since λU is in \mathcal{U} for every U in \mathcal{U} and every non-zero complex number λ , and a similar property holds in \mathcal{V} , we now see that $(E, \mathfrak{I} \vee \mathfrak{I}')^* = \cup \{ (U \cap V)^\circ : U \in \mathcal{U}, V \in \mathcal{V} \}$
 $= \cup \{ U^\circ + V^\circ : U \in \mathcal{U}, V \in \mathcal{V} \} = \cup \{ U^\circ : U \in \mathcal{U} \} + \cup \{ V^\circ : V \in \mathcal{V} \}$
 $= (E, \mathfrak{I})^* + (E, \mathfrak{I}')^*.$

2.1.7 Theorem. Let A be a Banach algebra with bounded approximate identity $\{e_\alpha\}$. Then (i) $(A, \beta^1)^* = A^* \cdot A$; (ii) $(A, \beta^2)^* = A \cdot A^*$; and (iii) $(A, \beta)^* = A^* \cdot A + A \cdot A^* = \{f \in A^* : \lim \|e_\alpha \cdot f + f \cdot e_\alpha - e_\alpha \cdot f \cdot e_\alpha - f\| = 0\}$. Moreover, each of these subspaces is norm-closed.

Proof. Since the proofs for (i) and (ii) are similar, we only prove (i). Clearly $A^* \cdot A \subset (A, \beta^1)^*$. Let g be an element of $(A, \beta^1)^*$. There exists an element a in A such that $|g(x)| \leq \|ax\|$ for all x in A ; then $\|g \cdot e_\alpha - g\| = \sup\{|g(e_\alpha x - x)| : \|x\| \leq 1\} \leq \sup\|ae_\alpha x - ax\| : \|x\| \leq 1\} \leq \|ae_\alpha - a\|$, and the last term of the inequality converges to 0. From [17] we know that $A^* \cdot A = \{g \in A^* : \lim \|g \cdot e_\alpha - g\| = 0\}$;

hence $(A, \beta^1) = A^* \cdot A$ and is easily seen to be norm-closed.

To prove (iii), we note that (i) and (ii) in conjunction with Lemma 2.1.6 shows that $(A, \beta)^* = (A, \beta^1)^* + (A, \beta^2)^* = A^* \cdot A + A \cdot A^*$.

Let $W = \{f \in A^* : \lim \|e_\alpha \cdot f + f \cdot e_\alpha - e_\alpha \cdot f \cdot e_\alpha - f\| = 0\}$. Then it is easily shown that W is norm-closed and that $A \cdot A^* + A^* \cdot A \subset W$. It remains to show that $W \subset (A, \beta)^*$.

We present the proof given by D. C. Taylor for a B^* -algebra [26], but supply additional argument in the one step at which he uses involution and the B^* -condition on the norm.

Let f be in W , and assume that $\|f\| \leq 1$. By induction we can choose $\{e_{\alpha_n}\}$, a sequence of elements of the bounded approximate identity $\{e_\alpha\}$, such that (i) $\alpha_{n+1} > \alpha_n$; (ii) $\|e_{\alpha_{n+1}} \cdot f + f \cdot e_{\alpha_{n+1}} - e_{\alpha_{n+1}} \cdot f \cdot e_{\alpha_{n+1}} - (e_{\alpha_n} \cdot f + f \cdot e_{\alpha_n} - e_{\alpha_n} \cdot f \cdot e_{\alpha_n})\| \leq 1/4^{n+1}$; (iii) $\|e_{\alpha_k} - e_{\alpha_k} e_{\alpha_{n+1}}\| \leq 1/9 \cdot 4^n$, $k = 1, 2, \dots, n$; and (iv) $\|e_{\alpha_k} - e_{\alpha_{n+1}} e_{\alpha_k}\| \leq 1/9 \cdot 4^n$, $k = 1, 2, \dots, n$.

Let $\{d_k\}$ be a sequence in A defined by $d_{5k-4} = (3/2^{k+1}) e_{\alpha_k}$,

$$d_{5k-3} = e_{\alpha_k} - e_{\alpha_k} e_{\alpha_{k+1}}, \quad d_{5k-2} = e_{\alpha_k} - e_{\alpha_{k+1}} e_{\alpha_k},$$

$$d_{5k-1} = e_{\alpha_k} - e_{\alpha_k} e_{\alpha_{k+2}}, \quad \text{and} \quad d_{5k} = e_{\alpha_k} - e_{\alpha_{k+2}} e_{\alpha_k}. \quad \text{Note that}$$

$\{d_k\}$ is a sequence in A with the property that

$\sum_{k=1}^{\infty} \|d_k\| < \infty$. Using involution and the B^* -condition on the

norm, Taylor proves that there exists in a B^* -algebra an hermetian element a and sequences $\{b_k\}$, and $\{c_k\}$ such that $\max\{\|b_k\|^2, \|c_k\|^2\} \leq \|d_k\|$ and $d_k = ab_k = c_k a$ for each k .

We circumvent this argument by defining an isometric A -module $\ell_1(A)$, consisting of all sequences $\{a_k\}$ in A with the property that $\sum_{k=1}^{\infty} \|a_k\| < \infty$, and by proving that $A \cdot \ell_1(A) = \ell_1(A) = \ell_1(A) \cdot A$; whence for $\{d_k\}$ in $\ell_1(A)$, there exist elements a and b in A and sequences $\{b_k\}$ and $\{c_k\}$ in $\ell_1(A)$ such that $d_k = ab_k = c_k b$ for each k and $\sum_{k=1}^{\infty} \|b_k\| < \infty$ and $\sum_{k=1}^{\infty} \|c_k\| < \infty$. With just these requirements the remainder of Taylor's argument is valid.

Let $\ell_1(A)$ denote all sequences $\{a_k\}$ in A with the property that $\sum_{k=1}^{\infty} \|a_k\| < \infty$; then $\ell_1(A)$, under coordinate-wise operations and norm defined by $\|\{a_k\}\|_1 = \sum_{k=1}^{\infty} \|a_k\|$, is a Banach space. For b in A and $\{a_k\}$ in $\ell_1(A)$ by defining $b \cdot \{a_k\}$ to be $\{ba_k\}$ and $\{a_k\} \cdot b$ to be $\{a_k b\}$, we see that $\ell_1(A)$ is an isometric left and an isometric right A -module. To show that $A \cdot \ell_1(A) = A = \ell_1(A) \cdot A$ it suffices by [17] to show that for each $\{a_k\}$ in $\ell_1(A)$ that $\lim \| \{a_k\} - e_{\alpha} \cdot \{a_k\} \|_1 = 0 = \lim \| \{a_k\} - \{a_k\} \cdot e_{\alpha} \|_1$. For $\epsilon > 0$ there is a positive integer N such that $\sum_{k=N+1}^{\infty} \|a_k\| < \epsilon/4$ and there is an α_0 such that for $\alpha > \alpha_0$, $\max\{\|e_{\alpha} a_k - a_k\|, \|a_k e_{\alpha} - a_k\|\} \leq \epsilon/2N$, for $n=1, 2, \dots, N$. Therefore, for $\alpha > \alpha_0$ we have $\max\{\|\{a_k\} - \{a_k\} \cdot e_{\alpha}\|_1, \|\{a_k\} - e_{\alpha} \cdot \{a_k\}\|_1\}$

$$= \max \{ \sum_{k=1}^N \|a_k - a_k e_\alpha\| + \sum_{k=N+1}^\infty \|a_k - a_k e_\alpha\|, \sum_{k=1}^N \|a_k - e_\alpha a_k\| + \sum_{k=N+1}^\infty \|a_k - e_\alpha a_k\| \} \leq N(\epsilon/2N) + 2 \sum_{k=N+1}^\infty \|a_k\| < \epsilon.$$

Returning now to the remainder of Taylor's proof, we have $d_k = ab_k = c_k b$ for each k and $\sum_{k=1}^\infty \|b_k\| < \infty$ and $\sum_{k=1}^\infty \|c_k\| < \infty$. Let $V = \{x \in A : \|xa\| < \delta \text{ and } \|bx\| < \delta\}$, where $\delta = 1/4 \cdot (\sum_{k=1}^\infty (\|b_k\| + \|c_k\|))$. Then if $|f(V)| \leq 1$, we have that f is an element of $(A, \beta)^*$.

For convenience of notation let $T_n(f)$ denote $e_{\alpha_n} \cdot f + f \cdot e_{\alpha_n} - e_{\alpha_n} \cdot f \cdot e_{\alpha_n}$, and let $S_n(x)$ denote $e_{\alpha_n} x + x e_{\alpha_n} - e_{\alpha_n} x e_{\alpha_n}$. The following computations are easily checked: (i) $|T_k(f)(x - S_{k+2}(x))| \leq 2\delta \|b_{5k}\| + 4\delta \|c_{5k-1}\|$; (ii) $\|S_k(x)\| \leq 2^{k+1}\delta \|c_{5k-4}\|/3 + 2^{k+2}\delta \|b_{5k-4}\|/3$; and (iii) $|T_{k+1}(f)(x - S_{k+2}(x))| \leq 2\delta \|b_{5(k+1)-2}\| + 4\delta \|c_{5(k+1)-3}\|$.

Since $f = T_1(f) + \sum_{k=1}^\infty (T_{k+1}(f) - T_k(f))$, we have this result for x in V : $|f(x)| \leq |T_1(f)(x)| + \sum_{k=1}^\infty |(T_{k+1}(f) - T_k(f))(x)| \leq |f(S_1(x))| + \sum_{k=1}^\infty [(T_{k+1}(f) - T_k(f))(x - S_{k+2}(x) + S_{k+2}(x))] \leq \|S_1(x)\| + \sum_{k=1}^\infty |T_{k+1}(f)(x - S_{k+2}(x))| + \sum_{k=1}^\infty |T_k(f)(x - S_{k+2}(x))| + \sum_{k=1}^\infty |(T_{k+1}(f) - T_k(f))(S_{k+2}(x))| \leq 2^3\delta \|b_1\|/3 + 2^2\delta \|c_1\|/3 + \sum_{k=1}^\infty (2\delta \|b_{5k+3}\| + 4\delta \|c_{5k+2}\| + \sum_{k=1}^\infty (2\delta \|b_{5k}\| + 4\delta \|c_{5k-1}\|) + \sum_{k=1}^\infty 4^{-k-1} (2^{k+4}\delta \|b_{5(k+2)-4}\|/3 + 2^{k+3}\delta \|c_{5(k+2)-4}\|/3) \leq 4\delta (\sum_{k=1}^\infty \|b_k\| + \sum_{k=1}^\infty \|c_k\|)$. Consequently, $|f(V)| \leq 1$, and f is in $(A, \beta)^*$.

The topological duals of the \mathcal{A} family remain to be computed. For the remainder of this section we assume that A has a bounded approximate identity and that the \mathcal{A} topologies are Hausdorff.

2.1.8 Theorem. $(A, \mathcal{A})^* = (A, \mathcal{A}^1)^* = (A, \mathcal{A}^2)^* = \langle P \rangle$.

Proof. Since $|\theta(a)|^2 \leq \|\theta\| \theta(a*a)$ and $\theta(a) = \overline{\theta(a^*)}$ for θ in P and a in A [7, p. 23], we have that $|\theta(a)|^2 \leq \|\theta\| \min\{\theta(a*a), \theta(aa^*)\}$, and so $\langle P \rangle \subseteq \bigcap_{\theta \in P} (A, \mathcal{A}^1)^* \cap (A, \mathcal{A}^2)^* \cap (A, \mathcal{A})^*$. For a in A define $|a| = \sup\{\theta(a*a)^{\frac{1}{2}} : \theta \in P, \|\theta\| \leq 1\}$. Since we are assuming that P separates points of A , the mapping $a \mapsto |a|$ is a norm on A . Let B denote the completion of A with respect to this norm. The space B is a B^* -algebra, and the natural injection $\tau: A \rightarrow B$ is norm-continuous [7, p. 41]. Furthermore, for each θ in P there exists a unique continuous linear functional $\bar{\theta}$ on B such that $\bar{\theta}(b*b) \geq 0$ for each b in B and such that $\bar{\theta} \circ \tau = \theta$ and the norm of $\bar{\theta}$ in B^* , denoted $|\bar{\theta}|$, equals $\|\theta\|$ by [7, p. 42].

Let f be an element of $(A, \mathcal{A}^1)^*$; then there exists θ in P such that $|f(x)| \leq \theta(x*x)^{\frac{1}{2}}$ for all x in A . Define a functional g on τA by $g(\tau a) = f(a)$. The following inequalities show that g is continuous on τA with respect to the norm on B : $|g(\tau a)|^2 = |f(a)|^2 \leq \theta(a*a) = \bar{\theta}((\tau a)*\tau a) \leq |\bar{\theta}| |\tau a|^2 = \|\theta\| |\tau a|^2$. Consequently, g extends uniquely to \bar{g} in B^* ; furthermore there are

functionals $\varphi_1, \varphi_2, \varphi_3$, and φ_4 on B such that $\varphi_i(b^*b) \geq 0$ for all b in B , $i=1,2,3,4$, and $\bar{g} = \varphi_1 - \varphi_2 + i\varphi_3 - i\varphi_4$ [7 ,p.40]. Note that $\varphi_i \circ \tau$ is in P for $i=1,2,3,4$ [,p.42]. So for each a in A we have that $f(a) = \bar{g}(\tau a) = g(\tau a)$; hence f is in $\langle P \rangle$. Similarly, $(A, \mathcal{J}^2)^* = \langle P \rangle$. Since $\mathcal{J} = \mathcal{J}^1 \vee \mathcal{J}^2$, we also have that $(A, \mathcal{J})^* = \langle P \rangle$ by Lemma 2.1.6. Hence the theorem is proved.

2.1.9 Corollary. With the notation as in the theorem, these are equivalent: (i) $\sigma(A, A^*) \leq \mathcal{J} \leq \tau(A, A^*)$; and (ii) A is complete in the B^* -norm $a \mapsto |a|$.

Proof. Let B denote the completion of A with respect to the norm $a \mapsto |a|$ on A . To prove (i) implies (ii), let $\tau: A \rightarrow B$ be the natural injection. Then since $A^* = (A, \mathcal{J})^* = \langle P \rangle$ by the Mackey-Arens theorem, since B^* is the linear span of the set $P' = \{\varphi \in B^* : \varphi(b^*b) \geq 0, \forall b \in B\}$ [7 ,p.40] and the mapping $\theta \mapsto \bar{\theta}$ from P to P' as in the theorem is a bijection [7 ,p.42], the adjoint map $\tau^*: B^* \rightarrow A^*$ is surjective. By [8 ,p.521] the subspace τA is closed in B ; since τA is dense, $\tau A = B$.

Conversely, with B defined as above the sets A and B are equal, and the identity map $I: A \rightarrow B$ is a homeomorphism by the Open Mapping Theorem. Therefore, A^* and B^* are equal as subsets of A' . Let $P' = \{\varphi \in A' : \varphi(x^*x) \geq 0, \forall x \in A\}$. By [28] we have that $P' = P$ in A' ; and since A and B

are homeomorphic, each element of P' is continuous on B . Hence $\langle P' \rangle = B^*$ since B is a B^* -algebra. Therefore, $(A, \mathcal{J})^* = \langle P \rangle = \langle P' \rangle = B^* = A^*$. Applying the Mackey-Arens Theorem, we have that $\sigma(A, A^*) \leq \mathcal{J} \leq \tau(A, A^*)$.

Section 2. Relationships among these Topologies.

In this section we investigate some of the basic relationships. This subject is investigated more fully in Chapter V.

2.2.1 Theorem. Let A be a Banach algebra with bounded approximate identity. We have the following equivalences:

(i) $\mu^1 \leq \beta^2$ if and only if $(A, \beta^1)^* = A^*$; and (ii) $\mu^2 \leq \beta^2$ if and only if $(A, \beta^2)^* = A^*$.

Proof. Since (i) and (ii) have similar proofs, we only prove the first assertion. If $\mu^1 \leq \beta^1$, then by Theorem 2.1.4 we have that $\sigma(A, A^*) \leq \mu^1 \leq \beta^1 \leq \tau(A, A^*)$; hence $(A, \beta^1)^* = A^*$ by the Mackey-Arens Theorem.

Conversely, if $(A, \beta^1)^* = A^*$, for any element f in A^* there exist elements a in A and g in A^* such that $f = g \cdot a$ by Theorem 2.1.7. Since the set $\{x \in A : \|ax\| \leq 1/(\|g\|+1)\}$ is contained in the set $\{x \in A : \|f \cdot x\| \leq 1\}$, we have that $\mu^1 \leq \beta^1$.

2.2.2 Theorem. If A is a Banach $*$ -algebra with bounded approximate identity, then (i) $\mu^1 \leq \beta^1$ if and only if $\mu^2 \leq \beta^2$; and (ii) both $\mu^1 \leq \beta^2$ and $\mu^2 \leq \beta^1$ if A is B^* .

Proof. If A has an involution, $(A, \beta^2)^* = A^*$ if and only if $(A, \beta^1)^* = A^*$ in virtue of Theorem 2.1.7 and the relation $(f \cdot a)^* = a^* \cdot f^*$ for f in A^* and a in A .

To prove (ii), let θ be an element of A^* such that $\theta(x^*x) \geq 0$ for every x in A . By [26] there exist elements a in A and φ in A^* such that $\varphi(x^*x) \geq 0$ for all x in A and $a \cdot \varphi \cdot a^* = \theta$. Then the set $\{x \in A : \|xa\| \leq (\|\varphi\| + 1)^{-\frac{1}{2}}\}$ is contained in the set $\{x \in A : \theta(x^*x) \leq 1\}$; therefore $\mu^1 \leq \beta^2$. Moreover, the set $\{x \in A : \|a^*x\| \leq (\|\varphi\| + 1)^{-\frac{1}{2}}\}$ is contained in the set $\{x \in A : \theta(x^*x) \leq 1\}$; therefore $\mu^2 \leq \beta^1$.

In general, the pairs of topologies μ^1 and β^2 , μ^2 and β^1 , and μ^1 and μ^2 -- are not comparable. Examples can be found in $B(H)$, where H is a separable Hilbert space. They are based on classical examples that involution is not continuous in the strong operator topology, that is, the locally convex Hausdorff topology generated by the seminorms $T \mapsto \|Tx\|$ for x in H . The reader is referred to [6, p.30] and [15, p.442].

2.2.3 Theorem. If A is a B^* -algebra, then we have the

following relationships: (i) $\sigma(A, A^*) \leq \mu^2 \leq \rho^1 \leq \beta^2 \leq \tau(A, A^*)$; (ii) $\sigma(A, A^*) \leq \mu^1 \leq \rho^2 \leq \beta^1 \leq \tau(A, A^*)$; and (iii) $\sigma(A, A^*) \leq \mu \leq \rho \leq \beta \leq \tau(A, A^*)$.

Proof. The proof of (i) and (ii) are similar, and (iii) follows immediately from them; therefore we only prove (i). Note that A has a bounded approximate identity by [7, p.15]. We have that $\rho^1 \leq \beta^2 \leq \tau(A, A^*)$ and $\sigma(A, A^*) \leq \mu^2$ by Theorems 2.1.4 and 2.2.2.

Since A is a B^* -algebra, for f in A^* there exist elements θ_i in A^* and complex numbers λ_i ($i=1,2,3,4$) with the properties that $\theta_i(x^*x) > 0$ for all x in A and $f = \sum_{i=1}^4 \lambda_i \theta_i$. For each θ_i and for each x in A we have by the Cauchy-Schwartz inequality that $\|x \cdot \theta_i\| = \sup\{|\theta_i(yx)| : y \in A, \|y\| \leq 1\} \leq \sup\{\theta_i(yy^*)^{\frac{1}{2}} \theta_i(x^*x)^{\frac{1}{2}} : y \in A, \|y\| \leq 1\} \leq \|\theta_i\|^{\frac{1}{2}} \theta_i(x^*x)^{\frac{1}{2}}$. Therefore, $\|x \cdot f\| \leq \sum_{i=1}^4 |\lambda_i| \|x \cdot \theta_i\| \leq \sum_{i=1}^4 |\lambda_i| \|\theta_i\|^{\frac{1}{2}} \theta_i(x^*x)^{\frac{1}{2}}$. Consequently, $\mu^2 \leq \rho^1$, and the first assertion is proved.

CHAPTER III

ARENS MULTIPLICATIONS ON THE BIDUALS OF BANACH ALGEBRAS

In this chapter we are concerned with Arens multiplications on A^{**} with respect to A and A^* . The topologies defined in Chapter II, under some circumstances, have extensions to locally convex topologies on A^{**} ; the relationships of these extensions to $\sigma(A^{**}, A^*)$ and $\tau(A^{**}, A^*)$ are interconnected with algebraic properties of the Arens multiplications. These relationships are considered in the first section. In the second, we are concerned specifically with Arens multiplications on the biduals of B^* and AB^* -algebras. The third section consists of various examples.

Section 1. Definitions of Arens Multiplications and of the μ Topologies.

We specialize Arens' definitions in [2] in the following way.

3.1.1 Definition. Let A be a Banach algebra, and let D be a subspace of A^* such that $D \cdot A$ (resp., $A \cdot D$) is a subset of D . Then for F in D' and f in D define an element $F \cdot f$ (resp., $F:f$) of A' by $F \cdot f(x) = F(f \cdot x)$,

(resp., $F:f(x) = F(x \cdot f)$) for all x in A . Let E be a subspace of D' such that $E \cdot D$ (resp., $E:D$) as a subset of D . Then for F and G in E define FG (resp., $F:G$) as an element of D' by $FG(f) = F(G \cdot f)$, (resp., $F:G(f) = F(G:f)$) for all f in D .

These bilinear operators defined on $E \times E$ to D' are called the Arens multiplications on E with respect to A and D . Provided that both multiplications are defined on E and that $FG = G:F$ for every F and G in E , these multiplications are said to be regular. If multiplication on A is commutative, the two Arens multiplications on E , if either is defined, coincide on E ; and the assertion that the Arens multiplications are regular is the assertion that this one multiplication is commutative.

Relations on the multiplications are summarized here for convenience and are used freely. If E contains the natural image of A in D' , no ambiguity occurs in the expressions FG and $G \cdot f$ where F and G are images of elements of A and f is an element of A^* ; therefore, we identify A with its natural imbedding in E and continue to denote elements of A with lower-case letters.

3.1.2 Lemma. For a in A ; f in D ; and F, G , and H in E we have the following relations if the terms that occur in each relation are well-defined: (i) $a:f = f \cdot a$;

- (ii) $aF = F:a$; (iii) $Fa = a:F$; (iv) $F \cdot (f \cdot a) = (F \cdot f) \cdot a$;
 (v) $F:(f:a) = (F:f):a$; (vi) $F \cdot (G \cdot f) = (FG) \cdot f$;
 (vii) $F:(G:f) = (F:G):f$; (viii) $(FG)H = F(GH)$; and
 (ix) $(F:G):H = F:(G:H)$.

Both Arens multiplications are defined on A^{**} with respect to A and A^* . Furthermore, A^{**} under either multiplication satisfies all the properties of a Banach algebra with the exception that if A^{**} has an identity, we can not show its norm must be 1. If A has a bounded approximate identity, then A^{**} is a Banach algebra as we shall see in Lemma 3.1.5.

For the Arens multiplications on A^{**} we also have the following continuity properties.

3.1.3 Lemma. Let F be an element of A^{**} , and define $R_F: A^{**} \rightarrow A^{**}$ by $R_F(G) = GF$ and ${}_F R: A^{**} \rightarrow A^{**}$ by ${}_F R(G) = G:F$. Then both R_F and ${}_F R$ are linear and $\sigma(A^{**}, A^*)$ -continuous.

Proof. These mappings are clearly linear. If $\{G_\alpha\}$ is a net converging to 0 in the $\sigma(A^{**}, A^*)$ topology, and if f is an element of A^* , then $0 = \lim G_\alpha(F \cdot f) = \lim G_\alpha F(f)$ and $0 = \lim G_\alpha(F:f) = \lim G_\alpha : F(f)$. Therefore, R_F and ${}_F R$ are $\sigma(A^{**}, A^*)$ -continuous.

The algebraic properties of these multiplications we consider are regularity and the existence of identities.

The existence of a $\sigma(A, A^*)$ -right (resp., -left) approximate identity for A is necessary and sufficient for there to exist an element I of A^{**} such that $F = FI$ (resp., $F = FI$) for every F in A^{**} [4]. However, for our purposes we assume that A has a bounded approximate identity.

3.1.4 Lemma. Let A have a bounded approximate identity $\{e_\alpha\}$. Then any $\sigma(A^{**}, A^*)$ -cluster point of $\{e_\alpha\}$ is a right identity for both multiplications in A^{**} .

Proof. Since $\{e_\alpha\}$ is a subset of the unit ball of A^{**} , a $\sigma(A^{**}, A^*)$ -compact set, there exists at least one cluster point for this net. Let I be one such point, let f be a non-zero element of A^* , and let a be an element of A . The net $\{e_\alpha\}$ converges to I in the $\sigma(A^{**}, (A, \beta)^*)$ topology since this net is a bounded approximate identity for A and $(A, \beta)^* = A \cdot A^* + A^* \cdot A$ by Theorem 2.1.7. Therefore, for $\epsilon > 0$ there is an α_0 such that for $\alpha > \alpha_0$, $\max \{\|a - ae_\alpha\|, \|a - e_\alpha a\|\} < \epsilon/2 \cdot \|f\|$ and $\max \{|I(f \cdot a) - f \cdot a(e_\alpha)|, |I(a \cdot f) - a \cdot f(e_\alpha)|\} < \epsilon/2$. Then $|I \cdot f(a) - f(a)| \leq |I(f \cdot a) - f \cdot a(e_\alpha)| + |f \cdot a(e_\alpha) - f(a)| < \epsilon/2 + \|f\| \|ae_\alpha - a\| < \epsilon$ for $\alpha > \alpha_0$. Since ϵ is arbitrary, $I \cdot f = f$ for each f in A^* . A similar calculation shows that $I : f = f$ for each f in A^* . So for F in A^{**} we have that $FI = F = F : I$.

3.1.5 Lemma. Let A have a bounded approximate identity.

If there exists a left identity I_0 for either one of the multiplications on A^{**} , then I_0 is also a right identity for the same multiplication. Moreover, I_0 is the $\sigma(A^{**}, A^*)$ -limit of any bounded approximate identity for A and $\|I_0\| = 1$.

Proof. Let $\{e_\alpha\}$ be a bounded approximate identity for A , and let J be some $\sigma(A^{**}, A^*)$ -cluster point of $\{e_\alpha\}$. Assuming that I_0 is a left $(:)$ -identity, for instance, and applying Lemma 3.1.4, we have $J = I_0 : J = I_0$. Hence I_0 is the unique $\sigma(A^{**}, A^*)$ -cluster point of the net $\{e_\alpha\}$ in a $\sigma(A^{**}, A^*)$ -compact set, namely, the unit ball of A^{**} . Thus, I_0 is the $\sigma(A^{**}, A^*)$ -limit of the net $\{e_\alpha\}$. It now follows that $\|I_0\| \leq 1$; and since $\|I_0\| = \|I_0 : I_0\| \leq \|I_0\|^2$, we have that $\|I_0\| = 1$.

We note that the set Γ of $\sigma(A^{**}, A^*)$ -cluster points in A^{**} of a bounded approximate identity for A is a (non-empty) $\sigma(A^{**}, A^*)$ -closed subset of the unit ball of A^{**} . Thus it is a $\sigma(A^{**}, A^*)$ -compact semigroup under either Arens multiplication: in fact, by Lemma 3.1.4 the Arens multiplications agree on Γ and are equal to left-trivial multiplication (that is, $IJ = I$ for I and J in Γ).

Let us consider the set Γ^r of right identities for either of the Arens multiplication. Then Γ^r is a semigroup with respect to this multiplication, which again agrees with

the left-trivial. This semigroup is $\sigma(A^{**}, A^*)$ -closed. To see this, suppose that Γ^r is the set of right identities for the multiplication $(F, G) \rightarrow F:G$. If I is an element of Γ^r , then $I:f = f$ for every f in A^* . If this were not true, there would exist an element f in A^* and F in A^{**} such that $F(f) = F:I(f) = F(I:f) \neq F(f)$, a contradiction. Therefore, if J is the $\sigma(A^{**}, A^*)$ -limit of a net $\{I_\beta\}$ in Γ^r , then for each a in A and f in A^* , we have $J:f(a) = J(f:a) = \lim I_\beta(f:a) = \lim I_\beta:f(a) = f(a)$. Then since $J:f = f$ for all f in A^* , we see that $F:J = F$ for all F in A^{**} . Thus, J is an element of Γ^r . We do not know if $\Gamma^r = \Gamma$ or even if Γ^r must be a subset of the unit ball of A^{**} . Note that Lemma 3.1.5 gives a sufficient condition for equality of these sets and that the condition implies $\Gamma^r = \Gamma$ consists of exactly one point.

Theorem 3.1.7 below gives necessary and sufficient conditions for there to exist an identity for either Arens multiplication on the bidual of a Banach algebra with bounded approximate identity.

3.1.6 Definition. The topologies μ, μ^1 , and μ^2 have extensions to locally convex topologies on A^{**} defined by the seminorms $F \rightarrow \max\{\|F:f\|, \|F \cdot f\|\}$, $F \rightarrow \|F:f\|$, and $F \rightarrow \|F \cdot f\|$, respectively, for f in A^* and F in A^{**} . The extension topologies are denoted by μ_e, μ_e^1 , and μ_e^2 .

3.1.7 Theorem. Let A have a bounded approximate identity. The following are equivalent: (i) there exists an element I of A^{**} such that $IF = F = FI$ (resp., $I:F = F = F:I$ for each F in A^{**}); (ii) $A^* = (A, \beta^1)^*$ (resp., $(A, \beta^2)^*$); (iii) $\sigma(A^{**}, A^*) \leq \mu_e^2$ (resp., μ_e^1); and (iv) μ_e^2 (resp., μ_e^1) is Hausdorff.

Proof. To show (i) implies (ii), let F be in A^{**} , f in A^* , and $\{e_\alpha\}$ a bounded approximate identity for A . By Lemma 3.1.5 we have that I is the $\sigma(A^{**}, A^*)$ -limit of $\{e_\alpha\}$. So $\lim F(f \cdot e_\alpha) = \lim F \cdot f(e_\alpha) = I(F \cdot f) = IF(f) = F(f)$. Thus A^* is the $\sigma(A^*, A^{**})$ -closure of $A^* \cdot A$. Since $A^* \cdot A = (A, \beta^1)^*$ and is convex and norm-closed, $A^* = (A, \beta^1)^*$.

(ii) \Rightarrow (iii). Let $\{F_\alpha\}$ be a net in A^{**} converging to 0 in the μ_e^2 topology. Then for each f in A^* there are elements a in A and g in A^* such that $f = g \cdot a$; hence, $|F_\alpha(f)| = |F_\alpha(g \cdot a)| = |F_\alpha \cdot g(a)| \leq \|F_\alpha \cdot g\| \|a\|$ and this last term converges to 0. Therefore, the net $\{F_\alpha\}$ converges to 0 in the $\sigma(A^{**}, A^*)$ topology.

(iii) \Rightarrow (iv) is clear. (iv) \Rightarrow (i). Let $\{e_\alpha\}$ be a bounded approximate identity for A and I_0 be a $\sigma(A^{**}, A^*)$ -cluster point of $\{e_\alpha\}$. For f in A^* and a in A we have that $\|f \cdot a e_\alpha - f \cdot a\| \leq \|f\| \|a e_\alpha - a\|$ and the latter term converges to 0. So for F in A^{**} we see that $F \cdot f(a) = F(f \cdot a) = \lim F(f \cdot a e_\alpha) = \lim (F \cdot f) \cdot a(e_\alpha) = I_0(F \cdot (f \cdot a)) = (I_0 F) \cdot f(a)$. Thus, $(F - I_0 F) \cdot f = 0$. Since the topology μ_e^2

is Hausdorff by assumption, $F = I_0 F$. Moreover, I_0 is a right identity by Lemma 3.1.4.

The proofs for the other equivalences are similar.

Arens [2] has given several necessary and sufficient conditions for regularity on the bidual of a Banach algebra, one of which is $\sigma(A^{**}, A^*)$ -continuity of either multiplication on A^{**} in the second variable. If f is an element of A^* , let $L_f: A \rightarrow A^*$ be the mapping defined by $L_f(a) = f \cdot a$ for a in A . Hennefeld [11] and Gulick [10] have shown that the Arens multiplications on A^{**} are regular if and only if for each f in A^* the mapping L_f is weakly compact; that is, if and only if the set $\{L_f(a): a \in A, \|a\| \leq 1\}$ in A^* is relatively $\sigma(A^*, A^{**})$ -compact. This criterion appears in the next theorem.

3.1.8 Theorem. Let A have a bounded approximate identity. If S denotes the unit ball of A^{**} , the following are equivalent: (i) the Arens multiplications are regular on A^{**} ; (ii) the sets $S \cdot f$ and $S : f$ are $\sigma(A^*, A^{**})$ -compact for each f in A^* ; (iii) $\sigma(A^{**}, A^*) \leq \mu_e \leq \tau(A^{**}, A^*)$; and (iv) $\sigma(A^{**}, A^*)|_S \leq \mu_e|_S \leq \tau(A^{**}, A^*)|_S$; that is, the relative topologies on S are so ordered.

Proof. To see that (i) \Rightarrow (ii), let f be an element of A^* and let $\{G_\alpha : f\}$ be a net in $S : f$. By the

$\sigma(A^{**}, A^*)$ -compactness of S we choose a $\sigma(A^{**}, A^*)$ -cluster point G in S of the net $\{G_\alpha\}$. Then for each F in A^{**} we have that GF is a $\sigma(A^{**}, A^*)$ -cluster point of the net $\{G_\alpha F\}$ by Lemma 3.1.3. Since the Arens multiplications are regular by hypothesis, we also have that $GF = F:G$ and $G_\alpha F = F:G_\alpha$ for each α . Therefore, $F(G:f) = F:G(f)$ is a cluster point of the net $\{F:G_\alpha(f)\}$ for each F in A^{**} . Hence, $G:f$, an element of $S:f$, is a $\sigma(A^*, A^{**})$ -cluster point of the net $\{G_\alpha:f\}$. The set $S \cdot f$ is $\sigma(A^*, A^{**})$ -compact by an analogous argument.

(ii) \Rightarrow (iii). Let $\{F_\alpha\}$ be a net in A^{**} converging to 0 in the $\tau(A^{**}, A^*)$ topology and let f be an element of A^* . Then $\{F_\alpha\}$ converges to 0 uniformly on every absolutely convex $\sigma(A^*, A^{**})$ -compact set and, in particular, on the sets $S \cdot f$ and $S:f$. Hence for $\epsilon > 0$ there is an α_0 such that for $\alpha > \alpha_0$ we have $\max\{|F_\alpha(H \cdot f)|, |F_\alpha(H:f)|\} < \epsilon$ for every H in S . Restricting H to lie in the unit ball of A , we have that $\max\{\|F:f\|, \|F f\|\} \leq \epsilon$; that is, the net $\{F_\alpha\}$ converges to 0 in the μ_e topology.

We now show that $\sigma(A^{**}, A^*) \leq \mu_e$. Let I be a $\sigma(A^{**}, A^*)$ -cluster point of a bounded approximate identity for A . Then $FI = F$ for every F in A^{**} by Lemma 3.1.4. If $IF = F$ for every F in A^{**} as well, we see that $\sigma(A^{**}, A^*) \leq \mu_e^2 \leq \mu_e$ by Theorem 3.1.7. Therefore, it suffices to prove that $IF = F$ for every F in A^{**} .

Assume, to the contrary, that there exists some F in A^{**} with $IF \neq F$. There exists a subnet $\{e_\beta\}$ of a bounded approximate identity for A such that I is the $\sigma(A^{**}, A^*)$ -limit of this subnet. Let g be an element of A^* such that $IF(g) \neq F(g)$. Since I is the $\sigma(A^{**}, A^*)$ -limit of $\{e_\beta\}$, we have $IF(g) = I(F \cdot g) = \lim F \cdot g(e_\beta) = \lim F(g \cdot e_\beta)$. Now, the net $\{g \cdot e_\beta\}$ is a subset of $S:g$, which is $\sigma(A^*, A^{**})$ -compact by hypothesis. Hence, there exists an element H in S so that $H:g$ is a $\sigma(A^*, A^{**})$ -cluster point of the net $\{g \cdot e_\beta\}$. Since this net converges to g in the $\sigma(A^*, A)$ topology, we see that $g = H:g$. Consequently, $F(g) = F(H:g)$ is a cluster point of the net $\{F(g \cdot e_\beta)\}$, but this net converges to $IF(g)$. Thus we have $F(g) = IF(g)$ in contradiction to the method in which g is chosen.

(iii) \Rightarrow (iv) is clear. Lastly, to show that (iv) \Rightarrow (i), let F and G be elements of A^{**} such that $\|F\| \leq 1$. There exists a net $\{a_\alpha\}$ in the unit ball of A which converges to F in the $\sigma(A^{**}, A^*)$ topology by Goldstine's Theorem. Then the convergence occurs with respect to the μ_e topology by hypothesis. In particular, for each f in A^* we have that $\max\{\|F \cdot f - a_\alpha \cdot f\|, \|F:f - f \cdot a_\alpha\|\}$ converges to 0. Therefore, $GF(f) = G(F \cdot f) = \lim G(a_\alpha \cdot f) = \lim G:f(a_\alpha) = F(G:f) = F:G(f)$ for each f in A^* ; thus $GF = F:G$.

The same theorem holds for μ_e^1 or μ_e^2 in place of μ_e . In fact, the proof of (ii) \Rightarrow (iii) shows that if $S:g$ is

$\sigma(A^*, A^{**})$ -compact for all g in A^* , then $\sigma(A^{**}, A^*) \leq \mu_e^2 \leq \mu$. Analogously, the $\sigma(A^*, A^{**})$ -compactness of $S \cdot g$ shows that $\sigma(A^{**}, A^*) \leq \mu_e^1 \leq \mu$. Finally in (iv) \Rightarrow (i) notice that the same reasoning applies for μ_e^1 or μ_e^2 in place of μ_e .

If A has an involution, there is a natural candidate for an involution on A^{**} .

3.1.9 Definition. Let A be a Banach $*$ -algebra. For f in A^* define $f^*(a) = \overline{f(a^*)}$, and for F in A^{**} define $F^*(f) = \overline{F(f^*)}$.

The operator $F \rightarrow F^*$ on A^{**} , which extends the involution on A , is isometric, conjugate-linear and idempotent. However, in the absence of regularity $(FG)^*$ can fail to be G^*F^* for F and G in A^{**} . A simple calculation shows that $(FG)^* = G^*:F^*$ and proves the following theorem.

3.1.10 Theorem. Let A be a Banach $*$ -algebra. The operator $F \rightarrow F^*$ on A^{**} is an involution if and only if the Arens multiplications on A^{**} are regular.

Section 2. B^* and AB^* -Algebras and Extension of the \mathscr{A} Topology.

In this section we define an AB^* -algebra, of which a B^* -algebra is an example. We give an extension of the \mathscr{A}

topology to A^{**} whose relationship to $\sigma(A^{**}, A^*)$ distinguishes B^* , AB^* and non- AB^* algebras. In the process of defining this extension we have an intermediate result that the Arens multiplications are regular on any AB^* -algebra with bounded approximate identity. The proof is an adaptation of one by Tomita [27] in which he shows the existence of an involution on the bidual of a B^* -algebra. We also include the essentials of his proof that this involution satisfies the B^* -condition.

3.2.1 Definition. Let A be a Banach $*$ -algebra. Then A is said to be an AB^* -algebra if the set $P = \{\theta \in A^* : \theta(x^*x) \geq 0, \forall x \in A\}$ separates points of A^{**} .

Any B^* -algebra is an AB^* -algebra with bounded approximate identity by [7, pp.15 and 40]. In the next section we give two examples of AB^* -algebras with bounded approximate identity which are not B^* under any equivalent norm.

~~The~~ The following theorem is used extensively and standardizes notation in this section.

3.2.2 Theorem. Let A be a Banach $*$ -algebra with bounded approximate identity. For each θ in P there exist a Hilbert space H_θ , a $*$ -representation $T_\theta : A \rightarrow B(H_\theta)$ and a cyclic vector h_θ in H_θ such that $\theta(a) = ((T_\theta(a)h_\theta, h_\theta))$ for

each a in A . Moreover, there exists a norm-continuous linear mapping $\bar{T}_\theta: A^{**} \rightarrow B(H_\theta)$ which extends T_θ and has the properties that $\bar{T}_\theta(F^*) = (\bar{T}_\theta(F))^*$ and $\bar{T}_\theta(FG) = \bar{T}_\theta(F) \circ \bar{T}_\theta(G)$ for every F and G in A^{**} .

Proof. For the existence of H_θ , T_θ , and h_θ refer to Theorem 4.5.14 of [16]. The $*$ -representation T_θ is norm-continuous by Theorem 4.5.4 of [16]. For x and y in H_θ define a linear functional $[x, y]$ on A by $[x, y](a) = ((T_\theta(a)x, y))$ for each a in A . Then $[x, y]$ is actually an element of A^* and $\|[x, y]\| \leq \|T_\theta\| \|x\| \|y\|$, where $\|x\|$ and $\|y\|$ denote the Hilbert-space norms.

Let F be an element of A^{**} . Define a mapping γ_F from $H_\theta \times H_\theta$ to the complex field by $\gamma_F(x, y) = F([x, y])$. Then γ_F is a bounded sesquilinear form on H . Consequently, there is a unique element $\bar{T}_\theta(F)$ in $B(H_\theta)$ such that $((\bar{T}_\theta(F)x, y)) = F([x, y])$ for each x and y in H and such that $\|\bar{T}_\theta(F)\| = \|\gamma_F\| \leq \|F\| \|T_\theta\|$ by [3, p.130].

The mapping $F \mapsto \bar{T}_\theta(F)$ is easily seen to be a norm-continuous linear mapping $\bar{T}_\theta: A^{**} \rightarrow B(H_\theta)$ which extends T_θ . By several easy calculations using the definition of \bar{T}_θ and using that T_θ is a $*$ -representation, we have $\bar{T}_\theta(FG) = \bar{T}_\theta(F) \circ \bar{T}_\theta(G)$ and $\bar{T}_\theta(F^*) = (\bar{T}_\theta(F))^*$ for each F and G in A^{**} .

3.2.3 Corollary. Each \bar{T}_θ for θ in P is $\sigma(A^{**}, A^*)$ -WOT

continuous.

Proof. Let $\{G_\alpha\}$ be a net in A^{**} which converges to 0 in the $\sigma(A^{**}, A^*)$ topology. Then for every x and y in H_θ we have that $((\bar{T}_\theta(G_\alpha)x, y)) = G_\alpha([x, y])$ converges to 0. Hence, the net $\{\bar{T}_\theta(G_\alpha)\}$ converges to 0 in the weak operator topology.

3.2.4 Lemma. Let A be a Banach $*$ -algebra with bounded approximate identity, let θ be an element of P , and let F and G be elements of A^{**} . Then (i) $F^*F(\theta) \geq 0$; (ii) $F^*G(\theta) = \overline{G^*F(\theta)}$; (iii) $|FG(\theta)|^2 \leq FF^*(\theta) \cdot G^*G(\theta)$; and (iv) the mapping $H \rightarrow H^*H(\theta)^{\frac{1}{2}}$ for H in A^{**} is a seminorm on A^{**} .

Proof. Let $T: A^{**} \rightarrow B(H)$ be the extended representation generated by θ and h the cyclic vector as in Theorem 3.2.2. Note that $\theta = [h, h]$. Therefore, $F^*F(\theta) = F^*F([h, h]) = \|T(F)h\|^2 \geq 0$. Furthermore, $F^*G(\theta) = F^*G([h, h]) = \overline{(T(G)^*T(F)h, h)} = \overline{G^*F(\theta)}$. Property (iii) now follows from (i) and (ii) by a standard argument [16, p.213]; and (iv) is an immediate consequence of (i), (ii), and (iii).

3.2.5 Theorem. The Arens multiplications are regular on an AB^* -algebra having a bounded approximate identity.

Proof. Let A be such an algebra. By Theorem 3.1.10 it suffices to show that $(FG)^* = G^*F^*$ for every F and G

in A^{**} . Suppose for some F and G this equality were false. Then since A is AB^* , there would exist an element θ in P such that $(FG)^*(\theta) \neq G^*F^*(\theta)$. Noting that $\theta^* = \theta$, we would then have $\overline{FG(\theta)} \neq G^*F^*(\theta)$ in contradiction to part (ii) of Lemma 3.2.4. Hence $(FG)^* = G^*F^*$ for each F and G in A^{**} .

The converse of this theorem is not true and that is shown in the next section.

For the remainder of this section let A denote a Banach $*$ -algebra with bounded approximate identity.

3.2.6 Definition. The seminorms $F \rightarrow \max\{FF^*(\theta)^{\frac{1}{2}}, F^*F(\theta)^{\frac{1}{2}}\}$ for F in A^{**} and θ in P define a locally convex topology on A^{**} called \mathcal{J}_e which clearly extends \mathcal{J} .

We prove that $\mathcal{J}_e \leq \tau(A^{**}, A^*)$ under the condition that \mathcal{J} is Hausdorff on A , or equivalently, that P separates points of A . The proof of this result, Theorem 3.2.10, is rather long and is divided into several lemmas, which are independently useful. We will assume that \mathcal{J} is Hausdorff through Theorem 3.2.10.

3.2.7 Lemma. Let F be an element of A^{**} and S be the Unit ball of A^{**} . We have the following: (i) if $R_F: A^{**} \rightarrow A^{**}$

is defined by $R_F(G) = GF$ and ${}_FR:A^{**} \rightarrow A^{**}$ by ${}_FR(G) = G:F$, then both R_F and ${}_FR$ are linear and $\sigma(A^{**}, A^*)$ -continuous; (ii) if $L_F:A^{**} \rightarrow A^{**}$ is defined by $L_F(G) = FG$ and ${}_FL:A^{**} \rightarrow A^{**}$ by ${}_FL(G) = F:G$, then both L_F and ${}_FL$ are linear and $\sigma(A^{**}, A^*) - \sigma(A^{**}, (A, \mathcal{J})^*)$ continuous; and (iii) the sets $S \cdot \theta$ and $S:\theta$ are absolutely convex and $\sigma(A^*, A^{**})$ -compact for each θ in P .

Proof. Part (i) is a restatement of Lemma 3.1.3 and is included here for the reader's convenience. To show $\sigma(A^{**}, (A, \mathcal{J})^*)$ -convergence, it suffices by Theorem 2.1.8 to show pointwise convergence on P . Let θ be an element of P . If $\{G_\alpha\}$ is a net in A^{**} converging to 0 in the $\sigma(A^{**}, A^*)$ topology, the net $\{G_\alpha^*\}$ converges to 0 in the same topology. Hence, $\lim FG_\alpha(\theta) = \lim \overline{G_\alpha^* F^*}(\theta) = 0$ and $\lim F:G_\alpha(\theta) = \lim \overline{G_\alpha^* F^*}(\theta) = 0$ in view of Lemma 3.2.4 and the equality $(F:G)^* = G^* F^*$. Thus, L_F and ${}_FL$ are $\sigma(A^{**}, A^*) - \sigma(A^{**}, (A, \mathcal{J})^*)$ continuous.

Let $\{F_\alpha \cdot \theta\}$ be a net in $S \cdot \theta$. Since S is $\sigma(A^{**}, A^*)$ -compact, there exists a $\sigma(A^{**}, A^*)$ -cluster point F in S for the net $\{F_\alpha\}$. For each G in A^{**} the element $L_G F$ is a $\sigma(A^{**}, (A, \mathcal{J})^*)$ -cluster point of the net $\{L_G F_\alpha\}$ by part (ii). In particular, $GF(\theta) = \lim GF_\beta(\theta)$, for some subnet $\{F_\beta\}$ of $\{F_\alpha\}$. Hence $F \cdot \theta$, which is an element of $S \cdot \theta$, is a $\sigma(A^*, A^{**})$ -cluster point of $\{F_\alpha \cdot \theta\}$.

The $\sigma(A^*, A^{**})$ -compactness of $S:\theta$ is shown similarly by means of the map G^L .

3.2.8 Lemma. Let θ be an element of P , let W be the set $\{G \in A^{**} : G * G(\theta) \leq 1\}$, and let U be the set $\{a \in A : \theta(a * a) \leq 1\}$. Then $W = U^{oo}$ in A^{**} .

Proof. We first prove that W is contained in U^{oo} . Let G be a non-zero element of W , choose f in U^o and $\epsilon > 0$, and set $\delta = \epsilon/3 \cdot \|G\|$. Then there exists an element a in A such that (i) $\|a\| \leq \|G\|$; (ii) $\|(G - a) \cdot \theta\| \leq \delta$; and (iii) $|G(f) - f(a)| < \epsilon/3$. To see this, we select a net $\{a_\alpha\}$ in A converging to G in the $\tau(A^{**}, A^*)$ topology such that $\|a_\alpha\| \leq \|G\|$ for each α by Goldstine's Theorem. Lemma 3.2.7 ensures us that $\{a_\alpha\}$ converges to G uniformly on the set $S:\theta$, where S denotes the unit ball of A^{**} . There is an α_0 such that for $\alpha > \alpha_0$ we have that $|(G - a_\alpha)(F:\theta)| < \delta$ for each F in S . Restricting F to lie in the unit ball of A , we see that $\|(G - a_\alpha) \cdot \theta\| \leq \delta$ for $\alpha > \alpha_0$. Furthermore, there is an α_1 such that $|G(f) - f(a_\alpha)| < \epsilon/3$ for $\alpha > \alpha_1$. Choose a equal to an a_α where $\alpha > \alpha_0$ and $\alpha > \alpha_1$; then a has the desired properties.

An application of Lemma 3.2.4 now yields the following:

$$|(G * G - a * a)(\theta)| \leq |G * (G - a)(\theta)| + |(G - a * a)(\theta)| = |G * (G \cdot \theta - a \cdot \theta)| + |a * (G - a)(\theta)| \leq 2\|G\| \|(G - a) \cdot \theta\| \leq 2\epsilon/3.$$
Since G is in W , by the choice of a we have that $\theta(a * a) \leq 2\epsilon/3$

$+ G*G(\theta) \leq 2\epsilon/3 + 1$; in particular, $\theta(a*a)^{\frac{1}{2}} \leq 2\epsilon/3 + 1$.
 Recall that since f is in U^0 , we have $|f(x)| \leq \theta(x*x)^{\frac{1}{2}}$
 for every x in $A[18, p.13]$. Hence, $|G(f)| < \epsilon/3$
 $+ |f(a)| \leq \epsilon/3 + \theta(a*a)^{\frac{1}{2}} \leq 1 + \epsilon$. As ϵ is arbitrary, we
 see that $|G(f)| \leq 1$. This argument proves W to be
 contained in U^{00} .

Since U is contained in W , for equality of W and
 U^{00} it suffices to show that W is absolutely convex and
 $\sigma(A^{**}, A^*)$ -closed. The set W is clearly absolutely convex.
 Let G in A^{**} be the $\sigma(A^{**}, A^*)$ -limit of a net $\{F_\alpha\}$ in
 W . Then by Lemma 3.2.7 we have that $G*G(\theta) = \lim G*F_\alpha(\theta)$.
 Since $\{F_\alpha\}$ is a subset of W , we see by Lemma 3.2.4 that
 $|G*F_\alpha(\theta)| \leq G*G(\theta)^{\frac{1}{2}} \cdot F_\alpha * F_\alpha(\theta)^{\frac{1}{2}} \leq G*G(\theta)^{\frac{1}{2}}$. Then
 $G*G(\theta) \leq \sup |G*F_\alpha(\theta)| \leq G*G(\theta)^{\frac{1}{2}}$; hence, $G*G(\theta) \leq 1$ and G
 is in W . We have now shown that W is $\sigma(A^{**}, A^*)$ -closed
 and have completed the proof.

3.2.9 Corollary. In the notation of the lemma W^0 (taken
 in A^*) is $\sigma(A^*, A^{**})$ -compact.

Proof. Let $\{f_\alpha\}$ be a net in W^0 and let U be
 as in the lemma. Then W^0 (in A^*) equals U^0 , which is
 $\sigma(A^*, A)$ -compact. Hence, there is an element f in W^0
 which is a $\sigma(A^*, A)$ -cluster point of the net $\{f_\alpha\}$. We
 show it is also a $\sigma(A^*, A^{**})$ -cluster point.

Let F be a non-zero element of A^{**} and $\epsilon > 0$. By the same argument as in the preceding proof we can select a in A with the properties: $\|a\| \leq \|F\|$ and $\|(F - a) \cdot \theta\| \leq (\epsilon^2)/18 \cdot \|F\|$. Then $F - a$ is an element of $(\epsilon/3)W$. Furthermore, for each α there is a $\beta > \alpha$ such that $|(f_\beta - f)(a)| < \epsilon/3$. Keeping in mind that $\{f_\alpha\} \cup \{f\}$ is a subset of W^0 , we have that for each α there exists some β so that $|F(f_\beta - f)| \leq |(F - a)(f_\beta)| + |(f_\beta - f)(a)| + |(a - F)(f)| < 3(\epsilon/3) = \epsilon$. This proves f to be a $\sigma(A^*, A^{**})$ -cluster point of the net $\{f_\alpha\}$.

3.2.10 Theorem. If A has a bounded approximate identity and \mathcal{J} is Hausdorff on A (or, equivalently, P separates points of A), then $\mathcal{J}_e \leq \tau(A^{**}, A^*)$.

Proof. It suffices to show that the polar (in A^*) of a basic \mathcal{J}_e -neighborhood of 0 is relatively $\sigma(A^*, A^{**})$ -compact. A base for \mathcal{J}_e -neighborhoods is all sets of the form $\{f \in A^{**} : FF^*(\theta) \leq 1 \text{ and } F^*F(\theta) \leq 1\}$ for θ in P . Let U be one such set; let $W = \{F \in A^{**} : F^*F(\theta) \leq 1\}$ and $W^* = \{F \in A^{**} : F^* \in W\}$. Then U equals the intersection of W and W^* . The set W is absolutely convex and $\sigma(A^{**}, A^*)$ -closed by Lemma 3.2.8; and since $F \rightarrow F^*$ is a $\sigma(A^{**}, A^*)$ -continuous mapping we also have that the absolutely convex set W^* is $\sigma(A^{**}, A^*)$ -closed. Therefore, U^0 is the $\sigma(A^*, A^{**})$ -closed absolutely convex hull of $W^0 \cup (W^*)^0$. The set W^0 in A^* is $\sigma(A^*, A^{**})$ -compact by Corollary 3.2.9; the set $(W^*)^0$ is

certainly closed in this topology; hence U° is contained in $W^\circ + (W^*)^\circ$. It is easy to see that if $(W^\circ)^*$ denotes the set $\{f \in A^*: f^* \in W^\circ\}$, we have $(W^\circ)^* = (W^*)^\circ$. Since $f \mapsto f^*$ is a $\sigma(A^*, A^{**})$ -continuous mapping, the set $(W^*)^\circ$ is also $\sigma(A^*, A^{**})$ -compact; therefore, so is $W^\circ + (W^*)^\circ$. This completes the proof.

We proceed to show that the relationship of \mathcal{J}_e to $\sigma(A^{**}, A^*)$ distinguishes B^* , AB^* , and non- AB^* algebras. Recall that in this section we are always assuming A to have a bounded approximate identity, but now we no longer require that the \mathcal{J} topology be Hausdorff on A .

In Lemma 3.1.4 we showed any $\sigma(A^{**}, A^*)$ -cluster point in A^{**} of a bounded approximate identity for A is a right identity for both Arens multiplications.

3.2.11 Lemma. Any $\sigma(A^{**}, A^*)$ -cluster point I in A^{**} of a bounded approximate identity for A has the property that $IF(\theta)$ for every F in A^{**} and θ in P .

Proof. Let I be such a point, F be an element of A^{**} , θ be an element of P , and $\{e_\beta\}$ be a subnet of the bounded approximate identity such that $IF(\theta)$ is the limit of $F \cdot \theta(e_\beta)$. If S denotes the unit ball of A^{**} , then $\{\theta \cdot e_\beta\}$ is a subset of S ; θ , a $\sigma(A^*, A^{**})$ -compact set by Lemma 3.2.7. Hence, there exists an element H of S such that $H \cdot \theta$ is a $\sigma(A^*, A^{**})$ -cluster point of the net $\{\theta \cdot e_\beta\}$.

Since θ is the $\sigma(A^*, A)$ -limit of $\{\theta \cdot e_\beta\}$, we also have that $\theta = H:\theta$. Consequently, $F(H:\theta)$ equals $F(\theta)$ and is a cluster point of $\{F(\theta \cdot e_\beta)\}$. By the choice of $\{e_\beta\}$, we then have $F(\theta) = IF(\theta)$.

A similar proof shows that such a cluster point I in A^{**} has the property that $I:F(\theta) = F(\theta)$ for every F in A^{**} and θ in P ; however, we do not need this result.

3.2.12 Lemma. For each F in A^{**} the following are equivalent: (i) $F^*F(\theta) = 0$ for every θ in P ; (ii) $F(\theta) = 0$ for every θ in P ; and (iii) $FF^*(\theta) = 0$ for every θ in P .

Proof. To show (i) \Rightarrow (ii), let I be a $\sigma(A^{**}, A^*)$ -cluster point of a bounded approximate identity for A . Then for each θ in P we have $|F(\theta)|^2 = |IF(\theta)|^2 \leq II^*(\theta) \cdot F^*F(\theta) = 0$ by hypothesis and Lemma 3.2.4. Hence (ii) follows.

(ii) \Rightarrow (i). We first show that $G:(G^* \cdot \theta)$ is an element of P for each G in A^{**} and θ in P . Clearly, $G:(G^* \cdot \theta)$ is an element of A^* . For x in A we have $G:(G^* \cdot \theta)(x^*x) = G(x^*x \cdot (G^* \cdot \theta)) = (Gx^*)(xG^*)(\theta) = (x^*:G)(xG^*)(\theta) = (xG^*)^*(xG^*)(\theta) \geq 0$.

Let I be a $\sigma(A^{**}, A^*)$ -cluster point of an hermetian bounded approximate identity for A ; then $I = I^*$ and $IF(\theta) = F(\theta)$ for each F in A^{**} and θ in P . Furthermore, I is a right multiplicative identity for either

multiplication on A^{**} by Lemma 3.1.4. For any complex number λ and θ in P we have that $(\lambda I + F^*) : [(\lambda I + F^*) * \theta]$ is an element of P , and by hypothesis evaluation of this functional at F gives 0. Expanding the resulting equation and using the hypothesis, we have the following:
 $\overline{\lambda F : F^*(\theta) + \lambda F F(\theta) + (F : F^*) F(\theta)} = 0$. Setting $\lambda = 1, -1, i$, and $-i$; multiplying the corresponding equations by 1, -1, i , and $-i$ respectively; and adding, we obtain $4F : F^*(\theta) = 0$.
 Thus, $F^* F(\theta) = \overline{(F^* F)^*(\theta)} = \overline{F^* : F(\theta)} = 0$.

To show that (iii) is equivalent to (i), note that $\overline{F(\theta)} = F^*(\theta)$ for each θ in P and apply the above arguments to F^* .

3.2.13 Lemma. If an element F is the $\tau(A^{**}, A^*)$ -limit of a net $\{a_\alpha\}$ in A with the property that $\|a_\alpha\| \leq \|F\|$ for each α , then F^* is the $\tau(A^{**}, A^*)$ -limit of the net $\{a_\alpha^*\}$ and $F^* F$ is the $\sigma(A^{**}, (A, \mathcal{J})^*)$ -limit of the net $\{a_\alpha^* a_\alpha\}$.

Proof. The mapping $f \rightarrow f^*$ in A^* is $\sigma(A^*, A^{**})$ -continuous. It now follows that $G \rightarrow G^*$ in A^{**} is $\tau(A^{**}, A^*)$ -continuous. Hence, we have the first assertion.

Let S denote the unit ball of A^{**} and let us assume that F is in S . The sets $S : \theta$ and $S \cdot \theta$ are $\sigma(A^*, A^{**})$ -compact for each θ in P by Lemma 3.2.7; so $\lim \|F \cdot \theta = a_\alpha \cdot \theta\| = 0$ and F^* is the uniform limit of $\{a_\alpha^*\}$

on $S \cdot \theta$. Let $\epsilon > 0$; then there is an α_0 such that for $\alpha > \alpha_0$ we have that $\|F \cdot \theta - a_\alpha \cdot \theta\| < \epsilon/2$ and $|F^* - a^*)(K \cdot \theta)| < \epsilon/2$ for each K in S . Hence, $|(F^*F - a_\alpha^*a_\alpha)(\theta)| \leq |(F^* - a^*)(F \cdot \theta)| + |a_\alpha^*(F \cdot \theta - a_\alpha \cdot \theta)| \leq \epsilon/2 + \|a_\alpha^*\| \| (F - a_\alpha) \cdot \theta \| < \epsilon$. The second assertion now follows.

3.2.14 Theorem. Let S denote the unit ball of A^{**} . Then the following are equivalent: (i) A is AB^* ; (ii) $\langle P \rangle$ is norm-dense in A^* ; (iii) \mathcal{J}_e is Hausdorff on A^{**} ; (iv) $\sigma(A^{**}, A^*)|_S \leq \mathcal{J}_e|_S$; and (v) there is a set of $\sigma(A^{**}, A^*)$ - WOT continuous cyclic $*$ -representations of A^{**} which separate points of A^{**} .

Proof. To show that (i) \Rightarrow (ii), note that the polar in A^{**} of $\langle P \rangle$ is zero by hypothesis. Therefore, A^* is the $\sigma(A^*, A^{**})$ -closed absolutely convex hull of P . Since the norm-closure of a convex set is the same as the $\sigma(A^*, A^{**})$ -closure, (ii) now follows.

(ii) \Rightarrow (iii). Let F be a non-zero element of S . There is an element f of A^* such that $F(f) > 0$, and there are elements θ_i in P and complex numbers λ_i ($i=1, 2, 3, 4$) such that $\|f - \sum_{i=1}^4 \lambda_i \theta_i\| < F(f)/2 \cdot \|F\|$. Therefore, $0 < F(f) < \sum_{i=1}^4 |\lambda_i| |F(\theta_i)| + \frac{1}{2} F(f)$; and so, $\max \{|F(\theta_i)| : i=1, 2, 3, 4\} > 0$. Now apply Lemma 3.2.12.

(iii) \Rightarrow (iv) Let $\{F_\alpha\}$ be a net in S converging to 0 in the \mathcal{J}_e topology. There exists a $\sigma(A^{**}, A^*)$ -cluster

point in S , say F , of this net. It suffices to show that F is 0. Either there exists a θ in P such that $F^*F(\theta) > 0$, or for each θ in P we have that $F^*F(\theta) = 0$. In the latter case Lemma 3.2.12 yields that $FF^*(\theta) = 0$ for each θ in P , and so by hypothesis F is 0. Assuming the former case holds, we reach a contradiction by means of the following argument. Since $\{F_\alpha\}$ converges to 0 in the \mathcal{L}_e topology, there is an α_0 such that $\alpha > \alpha_0$ implies $F_\alpha^*F_\alpha(\theta) < (1/4)F^*F(\theta)$. For any $\alpha > \alpha_0$ we have $|F^*F_\alpha(\theta)| \leq F^*F(\theta)^{\frac{1}{2}} \cdot F_\alpha^*F_\alpha(\theta)^{\frac{1}{2}} < \frac{1}{2}F^*F(\theta)$. By Lemma 3.2.7 we see that F^*F is a $\sigma(A^{**}, (A, \mathcal{L})^*)$ -cluster point of $\{F^*F_\alpha\}$. Hence $F^*F(\theta) \leq \limsup |F^*F_\alpha(\theta)| \leq \frac{1}{2}F^*F(\theta)$, which is a contradiction.

(iv) \Rightarrow (v). Applying Theorem 3.2.2 and its corollary to each θ in P , we generate a set of cyclic and $\sigma(A^{**}, A^*)$ -WOT continuous $*$ -representations of A^{**} . Moreover, for a non-zero element F of A^{**} there exists by hypothesis and Lemma 3.2.12 some θ in P such that $F(\theta) \neq 0$. If T denotes the representation generated by this θ and h its cyclic vector, then $T(F)$ is also unequal to 0 since $((T(F)h, h)) = F(\theta)$.

(v) \Rightarrow (i). We first show that for any $\sigma(A^{**}, A^*)$ -WOT continuous cyclic $*$ -representation $S: A^{**} \rightarrow B(H)$ with cyclic vector h this set, $\{S(a)h: a \in A\}$, is dense in H . Without loss of generality we may assume that the norm of h

is 1. Let x be any element of H and $\epsilon > 0$. There exists an element G of A^{**} such that $\|S(G)h - x\| < \frac{1}{2} \epsilon$. Choose a net $\{b_\alpha\}$ in A converging to G in the $\tau(A^{**}, A^*)$ topology with the property $\|b_\alpha\| \leq \|G\|$ for each α by Goldstine's Theorem. Then G^* is the $\tau(A^{**}, A^*)$ -limit of $\{b_\alpha^*\}$ and G^*G is the $\sigma(A^{**}, (A, \mathcal{J})^*)$ -limit of $\{b_\alpha^* b_\alpha\}$ by Lemma 3.2.13. The $\sigma(A^{**}, A^*)$ -WOT continuity of S implies that $S(G)$ and $S(G^*)$ are the WOT-limits of $\{S(b_\alpha)\}$ and $\{S(b_\alpha^*)\}$ respectively.

For y and z in H define a linear functional $[y, z]$ on A by $[y, z](a) = ((S(a)y, z))$; in particular, $[y, y](a^*a) \geq 0$ for each a in A . By [28] we have that $[y, y]$ is an element of P in A^* for each y in H . Since $4[y, z] = [y + z, y + z] - [y - z, y - z] + i[y + iz, y + iz] - i[y - iz, y - iz]$ for any y and z in H , the linear functional $[y, z]$ is also an element of A^* . Taking $y = S(G)h$ and $z = h$, we have $G^*([S(G)h, h]) = \lim [S(G)h, h] (b_\alpha^*) = \lim ((S(b_\alpha^*)S(G)h, h)) = ((S(G^*)S(G)h, h)) = \|S(G)h\|^2$.

Moreover, for a in A we have that $[S(G)h, h](a) = ((S(a)S(G)h, h)) = \lim ((S(b_\alpha)h, S(a^*)h)) = \lim ((S(ab_\alpha)h, h)) = \lim [h, h] \cdot a(b_\alpha) = G \cdot [h, h](a)$. Hence $G^*G([h, h]) = \|S(G)h\|^2$.

Since $[h, h]$ is an element of $(A, \mathcal{J})^*$ so that $G^*G([h, h]) = \lim [h, h] (b_\alpha^* b_\alpha)$, and since $[h, h](b_\alpha^* b_\alpha) =$

$\|S(b_\alpha)h\|^2$ for each α , we have $\|S(G)h\|^2 = \lim \|S(b_\alpha)h\|^2$. Using this equality and recalling that $S(G)$ is the WOT-limit of $S(b_\alpha)$, we see that $0 = \lim \|S(b_\alpha)h\|^2 - 2 \cdot \lim \operatorname{Re}((S(b_\alpha)h, S(G)h)) + \|S(G)h\|^2 = \lim \|S(b_\alpha)h - S(G)h\|^2$. Choose b_α such that $\|S(b_\alpha)h - S(G)h\| < \frac{1}{2}\epsilon$. Finally, for this b_α we have $\|S(b_\alpha)h - x\| \leq \|S(b_\alpha)h - S(G)h\| + \|S(G)h - x\| < \epsilon$. This proves that $\{S(a)h : a \in A\}$ is dense in H .

To show that (v) \Rightarrow (i), let F be a non-zero element of A^{**} and $S : A^{**} \rightarrow B(H)$ be a $\sigma(A^{**}, A^*)$ -WOT continuous cyclic $*$ -representation with cyclic vector h such that $S(F) \neq 0$. There are elements x and y in H and $\epsilon > 0$ such that $0 < \epsilon < |((S(F)x, y))|$. Choose a in A so that $\|S(a)h - x\| < \max \{\epsilon / (2\|S(F)\| \|y\|), \frac{1}{2}\|x\|\}$, and choose b in A so that $\|S(b)h - y\| < \epsilon / (2\|S(F)\| \|S(a)\|)$. Then since $\|h\| \leq 1$, we see that $|((S(F)x, y)) - ((S(Fa)h, S(b)h))| \leq |((S(F)x - S(Fa)h, y))| + |((S(Fa)h, y - S(b)h))| \leq \|S(F)\| \|x - S(a)h\| \|y\| + \|S(F)\| \|S(a)\| \|y - S(b)h\| < \epsilon$. Therefore, $((S(Fa)h, S(b)h)) \neq 0$.

Choose $\{c_\alpha\}$ to be a net in A having F as its $\tau(A^{**}, A^*)$ -limit and satisfying the property $\|c_\alpha\| \leq \|F\|$ for each α by Goldstine's Theorem. Then $F(a \cdot [h, h] \cdot b^*) = \lim a \cdot [h, h] \cdot b^*(c_\alpha) = \lim ((S(b^*c_\alpha a)h, h)) = \lim ((S(c_\alpha)S(a)h, S(b)h)) = ((S(F)S(a)h, S(b)h)) \neq 0$. Moreover, $4 a \cdot [h, h] \cdot b^* = (b^* + a^*) \cdot [h, h] \cdot (b^* + a^*)$

$-(b^* - a^*)^* \cdot [h, h] \cdot (b^* - a^*) + i(b^* + ia^*) \cdot [h, h] \cdot (b^* + ia^*)$
 $- i(b^* - ia^*)^* \cdot [h, h] \cdot (b^* - ia^*)^*$. Hence, for $\lambda = 1, -1, i$, or $-i$
 we see that $F((b^* + \lambda a^*)^* \cdot [h, h] \cdot (b^* + \lambda a^*)) \neq 0$. This
 inequality shows that there exists an element of P at
 which F is unequal to 0 and completes the proof of the
 theorem.

We remark that if A is an AB^* -algebra with bounded
 approximate identity, any $*$ -representation of its bidual
 must be norm-continuous. For such an algebra A the bidual
 A^{**} under either Arens multiplication is a Banach algebra
 with identity by Theorems 3.2.5 and 3.1.10 and by Lemmas
 3.1.4 and 3.1.5. Therefore, any $*$ -representation is norm
 continuous by [15, p. 241].

Part of Tomita's proof [27] that the bidual of a
 B^* -algebra is again B^* provides the second assertion of the
 next theorem, after which his complete proof is essentially
 given.

3.2.15 Theorem. Let A have a bounded approximate identity,
 and let S_0 denote the unit ball of A^* . Then (i) A is
 a B^* -algebra under an equivalent norm if and only if
 $\sigma(A^{**}, A^*) \leq \mathcal{J}_e$; and (ii) A is a B^* -algebra if and only if
 $S_0 = U\{x \in A : \theta(x^*x) \leq 1, \theta \in P \cap S_0\}^0$.

Proof of (i). Suppose A is a B^* -algebra under an

equivalent norm. Let γ be an element of $(A^{**}, \mathcal{J}_e)^*$, and let $\{F_\alpha\}$ be a net in A^{**} converging to 0 in the \mathcal{J}_e topology. Then applying Lemma 3.2.11 to choose an element I in A^{**} such that $IG(\theta) = G(\theta)$ for each G in A^{**} and applying Lemma 3.2.4, we have that $\lim |F_\alpha(\theta)| = \lim |IF(\theta)| \leq II^*(\theta)^{\frac{1}{2}} \cdot \lim F_\alpha^* F_\alpha(\theta)^{\frac{1}{2}} = 0$. Hence, \mathcal{J}_e -convergent nets converge pointwise on $\langle P \rangle$. Using this fact, the hypothesis and Corollary 2.6.4 of [7], we see that A^* is contained in $(A^{**}, \mathcal{J}_e)^*$. Consequently, $\sigma(A^{**}, A^*) \leq \mathcal{J}_e$ by the Mackey-Arens Theorem.

Conversely, suppose that $\sigma(A^{**}, A^*) \leq \mathcal{J}_e$. For f in A^* there exists some θ in P such that the set $U = \{F \in A^{**} : FF^*(\theta) \leq 1 \text{ and } F^*F(\theta) \leq 1\}$ is contained in the set $V = \{F \in A^{**} : |F(f)| \leq 1\}$. Then the linear functional f is an element of V^0 in A^* , which is contained in U^0 . Applying Lemma 3.2.8 we see that f is an element of $(A, \mathcal{J})^*$. Thus, $A^* = (A, \mathcal{J})^*$. Corollary 2.1.9 implies there exists a B^* -norm $x \mapsto |x|$ on A under which A is complete and for which the identity function $I: A \rightarrow (A, x \mapsto |x|)$ is continuous. An application of the Open Mapping Theorem concludes the proof.

Proof of (ii). Before we begin the proof we make the following observation. If W denotes the union $P \cap S_0$ of the sets $\{a \in A : \theta(a^*a) \leq 1\}^0$ for θ in $P \cap S_0$, and if U_θ

(for a specific θ) denotes the set $\{a \in A : \theta(a*a) \leq 1\}$, then $\theta(a*a)^{\frac{1}{2}} = \sup\{|f(a)| : f \in U_\theta^0\}$ for each a in A and θ in P . To see this, we note that for each f in U^0 we have $|f(x)| \leq 1$ whenever $\theta(x*x)^{\frac{1}{2}} \leq 1$ and so $|f(x)| \leq \theta(x*x)^{\frac{1}{2}}$ for every x in A by [18,p.13]. Consequently, $\sup\{|f(a)| : f \in U_\theta^0\} \leq \theta(a*a)^{\frac{1}{2}}$ for each a in A . By the Hahn-Banach Theorem for any point a in A there is a linear functional f such that $|f(x)| \leq \theta(x*x)^{\frac{1}{2}}$ for all x in A and $f(a) = \theta(a*a)^{\frac{1}{2}}$; this linear functional f is an element of U_θ^0 and, thus $\theta(a*a)^{\frac{1}{2}} = \sup\{|f(a)| : f \in U_\theta^0\}$.

Suppose A is a B^* -algebra; then $\|a\| = \sup\{\theta(a*a)^{\frac{1}{2}} : \theta \in P \cap S_0\}$ for each a in A by [7 ,p. 40]. By the remark above, we have $\|a\| = \sup\{|f(a)| : f \in W\}$ for each a in A . Thus W^0 (in A) is the unit ball of A and W^{00} (in A^*) is S_0 .

To show that $W = S_0$, it suffices to show that W is absolutely convex and $\sigma(A^*, A)$ -closed. We first prove convexity. Let f and g be elements of W and $0 < \lambda < 1$. Say f is an element of U_θ^0 and g is an element of U_φ^0 so that $|f(a)|^2 \leq \theta(a*a)$ and $|g(a)|^2 \leq \varphi(a*a)$ for each a in A . Set $\gamma = \lambda\theta + (1-\lambda)\varphi$; then γ is in $P \cap S_0$. The convexity in $(0, \infty)$ of the real function $x \mapsto x^2$ gives us for each a in A that $|\lambda f(a) + (1-\lambda)g(a)|^2 \leq (\lambda|f(a)|^2 + (1-\lambda)|g(a)|^2) \leq \gamma(a*a)$.

Consequently, $\lambda f + (1 - \lambda)g$ is an element of W . The absolute convexity of W now follows.

To prove that W is $\sigma(A^*, A)$ -closed, we let $\pi: (P \cap S_0) \times S_0 \rightarrow S_0$ be the projection on the second coordinate and let $W_1 = \{(\theta, f) \in (P \cap S_0) \times S_0 : f \in U_\theta^0\}$. Note that $P \cap S_0$ as well as S_0 is $\sigma(A^*, A)$ -compact; the product $(P \cap S_0) \times S_0$ is then compact in the product topology. Moreover, W_1 is closed in this topology: for if $\{(\theta_\alpha, f_\alpha)\}$ is a net in W_1 converging to (θ, g) in $(P \cap S_0) \times S_0$, then $g(a) = \lim f_\alpha(a)$ and $\theta(a^*a)^{\frac{1}{2}} = \lim \theta_\alpha(a^*a)^{\frac{1}{2}}$ for each a in A ; since $|f_\alpha(a)| \leq \theta_\alpha(a^*a)^{\frac{1}{2}}$ for each α , we have $|g(a)| = \lim |f_\alpha(a)| \leq \lim \theta_\alpha(a^*a)^{\frac{1}{2}} = \theta(a^*a)^{\frac{1}{2}}$; thus (θ, g) is an element of W_1 . Therefore, W_1 is compact in the product topology; and W , being the image of W_1 under the continuous mapping π is $\sigma(A^*, A)$ -compact. The assertion that $W = S_0$ is now proved.

Conversely, assume that $S_0 = W$. From our initial observation we have for each a in A that $\|a\| = \sup\{|f(a)| : f \in S_0\} = \sup\{|f(a)| : f \in W\} = \sup\{\theta(a^*a)^{\frac{1}{2}} : \theta \in P \cap S_0\} \leq \|a^*a\|^{\frac{1}{2}}$. Therefore, $\|a\| \leq \|a^*a\|^{\frac{1}{2}} \leq \|a\|$ for each element a of A ; so A is a B^* -algebra. This completes the proof of the theorem.

We now give the essentials of Tomita's proof that the bidual of a B^* -algebra is B^* .

3.2.16 Theorem. If C is a B^* -algebra, then C^{**} with the Arens multiplication $(F, G) \rightarrow FG$ is also B^* .

Proof. We note that C has a bounded approximate identity. Application of Theorems 3.1.10 and 3.2.5 gives us that $F \rightarrow F^*$ is an isometric involution on A^{**} . Since $C^* = \langle P \rangle$ by [7, p.40], then C^{**} has an identity by Lemmas 3.1.4 and 3.2.11. Part (ii) of Theorem 3.2.15 and Lemma 3.2.8 show that $S_0 = \bigcup \{F \in C^{**} : F^*F(\theta) \leq 1, \theta \in P \cap S_0\}^0$. Taking the polar in C^{**} of the set $\{F \in C^{**} : F^*F(\theta) \leq 1\}^0$ for any θ in P , we recover the set $\{F \in C^{**} : F^*F(\theta) \leq 1\}$ by Lemma 3.2.8. If S denotes the unit ball of C^{**} , then $S = \bigcap \{F \in C^{**} : F^*F(\theta) \leq 1, \theta \in P \cap S_0\}$. Using this equality and the fact that $G \rightarrow \sup \{G^*G(\theta)^{\frac{1}{2}} : \theta \in P \cap S_0\}$ is a norm, we have $\|F\| = \sup \{F^*F(\theta)^{\frac{1}{2}} : \theta \in P \cap S_0\} \leq \|F^*F\|^{\frac{1}{2}} \leq \|F\|$ for each F in C^{**} by [18, p.13]. Hence, C^{**} is a B^* -algebra.

Section 3. Examples.

Using a general method for constructing AB^* -algebras, we give two examples having bounded approximate identities, one commutative and one non-commutative, which are not B^* under any equivalent norm. Algebraic properties of AB^* -algebras are investigated with partial results. To characterize those Banach algebras for which the Arens multiplications on the bidual are regular remains unsolved.

We conclude this section with various examples on which to base conjectures concerning this problem.

3.3.1 Definition. Let $\{A_n\}_{n=1}^{\infty}$ be a family of Banach spaces. Let $c_0(\Sigma \oplus A_n)$ be the c_0 -direct sum: a typical element is a sequence (x_n) such that x_n is in A_n for each n and $\lim \|x_n\|_n = 0$, where $\|x_n\|_n$ denotes the norm of x_n in A_n . Let $\ell_1(\Sigma \oplus A_n)$ be the ℓ_1 -direct sum: a typical element is a sequence (y_n) such that y_n is in A_n for each n and $\sum_{n=1}^{\infty} \|y_n\|_n < \infty$.

For a further discussion of direct sums the reader is referred to [7, Ch. 10] and [16]. The proof of the next theorem is straightforward.

3.3.2 Theorem. For a family of Banach $*$ -algebras $\{A_n\}_{n=1}^{\infty}$ the c_0 -direct sum is a Banach $*$ -algebra under the following definitions for (a_n) and (b_n) in $c_0(\Sigma \oplus A_n)$ and any complex number λ : (i) $(a_n) + (b_n) = (a_n + b_n)$; (ii) $\lambda(a_n) = (\lambda a_n)$; (iii) $(a_n)(b_n) = (a_n b_n)$; (iv) $(a_n)^* = (a_n^*)$; and (v) $\|(a_n)\| = \sup \|a_n\|_n$.

3.3.3 Theorem. For a family of Banach spaces $\{B_n\}_{n=1}^{\infty}$ the ℓ_1 -direct sum is a Banach space under the following definitions for (a_n) and (b_n) in $\ell_1(\Sigma \oplus B_n)$ and any complex number λ : (i) $(a_n) + (b_n) = (a_n + b_n)$; (ii) $\lambda(a_n) = (\lambda a_n)$;

and (iii) $\|(a_n)\|_1 = \sum_{n=1}^{\infty} \|a_n\|_n$.

Moreover, if $\{A_n\}_{n=1}^{\infty}$ is a family of Banach algebras, then to each linear functional f in $(c_0(\Sigma \oplus A_n))^*$ there corresponds a unique element (g_n) in $\ell_1(\Sigma \oplus A_n^*)$ satisfying the following properties: (i) $f(a) = \sum_{n=1}^{\infty} g_n(a_n)$ for each $a = (a_n)$ in $c_0(\Sigma \oplus A_n)$; and (ii) if $\|f\| = \sup \{|f(a)| : a \in c_0(\Sigma \oplus A_n), \|a\| \leq 1\}$, then $\|f\| = \|(g_n)\|_1$.

Proof. Verification of the first assertion is straightforward. Let A denote the space $c_0(\Sigma \oplus A_n)$ and A^* the space $(c_0(\Sigma \oplus A_n))^*$ under the norm defined above. For each element x of A_k ($k=1,2,\dots$) define \bar{x} to be the element of A whose n -th coordinate is x if $n=k$ and $n \neq k$. Then the mapping $x \rightarrow \bar{x}$ from A_k to A is an isometric linear mapping. If f is an element of A^* , the complex-valued mapping g_k defined on A_k ($k=1,2,\dots$) by $g_k(x) = f(\bar{x})$ is an element of A_k^* . Moreover, if $a = (a_n)$ is an element of A , then $a = \sum_{n=1}^{\infty} \bar{a}_n$ and $f(a) = \sum_{n=1}^{\infty} f(\bar{a}_n) = \sum_{n=1}^{\infty} g_n(a_n)$.

Suppose $\sum_{n=1}^N \|g_n\|_n > \|f\|$ for some positive integer N . Choose ϵ such that $0 < \epsilon < \sum_{n=1}^N \|g_n\|_n - \|f\|$. For $n=1,2,\dots,N$ choose x_n in A_n such that $\|x_n\|_n \leq 1$, $g_n(x_n) > 0$, and $\|g_n\|_n - g_n(x_n) < \epsilon/N$. Then $\sum_{n=1}^N \bar{x}_n$ is an element in A of norm at most 1, and $\sum_{n=1}^N \|g_n\|_n < \epsilon + \sum_{n=1}^N g_n(x_n) = \epsilon + \sum_{n=1}^N f(\bar{x}_n) = \epsilon + f(\sum_{n=1}^N \bar{x}_n) \leq \epsilon + \|f\|$, a contradiction. Therefore,

$\sum_{n=1}^N \|g_n\|_n \leq \|f\|$ for every positive integer N ; thus (g_n) is an element of $\ell_1(\Sigma \oplus A_n^*)$ and $\|(g_n)\|_1 \leq \|f\|$. Since the reverse inequality is easily checked and the uniqueness of (g_n) is clear, the proof is thereby completed.

Some algebraic properties of AB^* -algebras are established in the next two theorems.

3.3.4 Theorem. Let $\{A_n\}_{n=1}^\infty$ be a family of AB^* -algebras, each of which has a bounded approximate identity. Then $c_0(\Sigma \oplus A_n)$ is also an AB^* -algebra with bounded approximate identity.

Proof. We first show that $c_0(\Sigma \oplus A_n)$ has a bounded approximate identity. For each positive integer n let $\{e(n, \gamma) : \gamma \in \Gamma_n\}$ be a bounded approximate identity for A_n . Then for any positive integer k and any choice function $c_k : \{1, 2, \dots, k\} \rightarrow \bigcup_{n=1}^k \Gamma_n$ such that $c_k(n)$ is in Γ_n let $I(k, c_k)$ be the element of $c_0(\Sigma \oplus A_n)$ whose n -th coordinate is $e(n, c_k(n))$ if $n = 1, 2, \dots, k$ and 0 otherwise. Define a partial ordering on the pairs (k, c_k) in the following way: $(k, c_k) \leq (k', c_{k'})$ if and only if $k \leq k'$ and $c_k(n) \leq c_{k'}(n)$ in the ordering on Γ_n for $n = 1, 2, \dots, k$. Then the collection of pairs (k, c_k) , denoted by Γ , is a directed set.

Moreover, the net $\{I(k, c_k) : (k, c_k) \in \Gamma\}$ is a bounded approximate identity for $c_0(\Sigma \oplus A_n)$. Note that for each pair (k, c_k) the norm of $I(k, c_k)$ is at most 1. Let

$x = (x_n)$ be an element of $c_0(\Sigma \oplus A_n)$ and $\epsilon > 0$. Then there is a positive integer N such that for $n > N$ the norm of x_n is at most $\epsilon/2$. For $n=1,2,\dots,N$ there exist $e(n, \gamma_n)$ in Γ_n such that $\|x_n - x_n e(n, \gamma_n)\|_n < \epsilon/2$. Define $c_N(n) = \gamma_n$ for $n=1,2,\dots,N$. Then $\|x - xI(N, c_N)\| \leq \max \{\|x_n - x_n e(n, \gamma_n)\|_n : n=1,\dots,N\} + \sup \{\|x_n\|_n : n > N\} \leq \epsilon$. Hence, $\{I(k, c_k) : (k, c_k) \in \Gamma\}$ is a right bounded approximate identity; it is also a left bounded approximate identity by a similar argument.

To show that $c_0(\Sigma \oplus A_n)$ is AB^* , it suffices by Theorem 3.2.14 that the linear span of $P = \{\theta \in \ell_1(\Sigma \oplus A_n^*) : \theta(x^*x) \geq 0, \forall x \in c_0(\Sigma \oplus A_n)\}$ is dense in $\ell_1(\Sigma \oplus A_n^*)$. Let $f = (f_n)$ be an element of $\ell_1(\Sigma \oplus A_n^*)$ and $\epsilon > 0$. There is a positive integer N such that $\sum_{n=N+1}^{\infty} \|f_n\|_n < \epsilon/2$. For each f_k in A_k^* ($k=1,2,\dots,N$), since A_k^* is AB^* , there are elements $\theta_{k,i}$ ($i=1,2,3,4$) in A_k^* with the property that $\theta_{k,i}(x_k^*x_k) \geq 0$ for every x_k in A_k and there are complex numbers $\lambda_{k,i}$ such that $\|f_k - \sum_{i=1}^4 \lambda_{k,i} \theta_{k,i}\|_k < \epsilon/2N$. Let $\bar{\theta}_{k,i}$ ($k=1,\dots,N; i=1,\dots,4$) denote the element of $\ell_1(\Sigma \oplus A_k^*)$ whose n -th coordinate is $\theta_{k,i}$ if $n=k$ and 0 if $n \neq k$. Note that for each $a = (a_n)$ in $c_0(\Sigma \oplus A_n)$ we have that $\bar{\theta}_{k,i}(a^*a) = \theta_{k,i}(a_k^*a_k) \geq 0$, ($k=1,\dots,N; i=1,\dots,4$). Then $\sum_{i=1}^4 \sum_{k=1}^N \lambda_{k,i} \bar{\theta}_{k,i}$ is an element of $\langle P \rangle$. Moreover, $\|f - \sum_{i=1}^4 \sum_{k=1}^N \lambda_{k,i} \bar{\theta}_{k,i}\| = \sum_{n=1}^N \|f_n - \sum_{i=1}^4 \lambda_{n,i} \theta_{n,i}\|_n + \sum_{n=N+1}^{\infty} \|f_n\|_n < \epsilon$ by the choice of N and of $\sum_{i=1}^4 \lambda_{n,i} \theta_{n,i}$

for $n=1, \dots, N$. The proof of the theorem is now concluded.

3.3.5 Theorem. If A is an AB^* -algebra, then any closed $*$ -subalgebra of A is also AB^* . If A also has a bounded approximate identity, then the algebra A_1 formed in the usual way by adjoining an identity to A is AB^* .

Proof. In this proof for a Banach $*$ -algebra C let P_C denote the set $\{\theta \in C^*: \theta(c^*c) \geq 0, \forall c \in C\}$. Let B be a closed $*$ -subalgebra of A ; then B is a Banach $*$ -algebra. If g is an element of B^* , by the Hahn-Banach Theorem there is an element \bar{g} in A^* such that $\bar{g}(b) = g(b)$ for every b in B . For $\epsilon > 0$ there exist elements θ_i in P_A and complex numbers λ_i ($i = 1, 2, 3, 4$) such that $\|\bar{g} - \sum_{i=1}^4 \lambda_i \theta_i\| < \epsilon$. Note that $\theta_i|_B$ is an element of P_B and $\|g - \sum_{i=1}^4 \lambda_i|_B\| < \epsilon$. Therefore, B is an AB^* -algebra by Theorem 3.2.14.

Consider A_1 as the set of elements of the form $x + \lambda e$, where e is the adjoined identity, x is an element of A , and λ is a complex number. Then A_1 , formed in the usual way, with algebraic operations and involution defined by $x = \lambda e \rightarrow x^* + \bar{\lambda}e$ is a Banach $*$ -algebra under the norm $\|x + \lambda e\| = \|x\| + |\lambda|$.

Let f be an element of $(A_1)^*$ and $\epsilon > 0$. Since $f|_A$ is an element of A^* , by hypothesis there exist θ_i in P_A and complex numbers λ_i ($i=1,2,3,4$) such that

$\|f|_A - \sum_{i=1}^4 \lambda_i \theta_i\| < \epsilon$. Define a functional $\bar{\theta}_i$ on A_1 by $\bar{\theta}_i(x + \lambda e) = \theta_i(x) + \lambda \|\theta_i\|$. Then $\|\bar{\theta}_i\| = \|\theta_i\|$ for each i , and each $\bar{\theta}_i$ is an element of $P_{(A_1)}$ by [7, p. 23]. Define a linear functional g on A_1 by $g(x + \lambda e) = \lambda(f(e) - \sum_{i=1}^4 \lambda_i \|\theta_i\|)$. Clearly g is an element of $\langle P_{(A_1)} \rangle$. Then $\|f - (\sum_{i=1}^4 \lambda_i \bar{\theta}_i + g)\| = \|f|_A - \sum_{i=1}^4 \lambda_i \theta_i\| < \epsilon$. Therefore, A_1 is an AB*-algebra by Theorem 3.2.14.

3.3.6 Lemma. Let A be a Banach *-algebra. If A is a B*-algebra under an equivalent norm, then there exists a constant λ such that $\|x\|^2 \leq \lambda \|x^*x\|$ for every x in A .

Proof. Let $x \mapsto |x|$ be a norm on A such that $|x^*x| = |x|^2$ for each x in A and for which the identity mapping $I: A \rightarrow (A, x \mapsto |x|)$ is a homeomorphism. There exist constants α and β such that $|x| \leq \alpha \|x\| \leq \beta |x|$ for every x in A . Set λ equal to β^2/α ; then $\|x\|^2 \leq (\beta^2/\alpha^2) |x| = (\beta^2/\alpha^2) |x^*x| \leq \lambda \|x^*x\|$ for each x in A .

A partial converse of this lemma is given by Yood in [30]. We now construct the promised examples of AB*-algebras.

3.3.7 Example. Let T be a locally compact Hausdorff space. A continuous (complex-valued) function f on T is said to vanish at infinity if for every $\epsilon > 0$ there is a compact set K in T such that $|f(t)| < \epsilon$ for every t in T that is not in K . Let $C_0(T)$ denote those continuous

functions on T which vanish at infinity. Under pointwise algebraic operations and norm defined by $\|f\|_\infty = \sup\{|f(t)| : t \in T\}$, $C_0(T)$ is a Banach algebra. Moreover, $C_0(T)$ is a B^* -algebra with involution defined by $f^*(t) = \overline{f(t)}$ for f in $C_0(T)$ and t in T .

For f in $C_0(T)$ and any positive integer n define $\|f\|_n = \|f\|_\infty + n \sup\{|f(t) - f(t')| : t, t' \in T\}$. Then $f \mapsto \|f\|_n$ is a norm under which $C_0(T)$ is a Banach $*$ -algebra; moreover, it is equivalent to the B^* -norm. By Theorem 3.3.4 we see that $c_0(\Sigma \oplus (C_0(T), f \mapsto \|f\|_n))$ is a commutative AB^* -algebra with bounded approximate identity. Furthermore, it is not B^* under any equivalent norm. Suppose that it were: then by Lemma 3.3.6 there would exist a constant λ such that $\|x\|^2 \leq \lambda \|x^*x\|$ for each x in the c_0 -direct sum. Choose an integer $N > \lambda$, two distinct points t and t' of T , and a non-negative function h in $C_0(T)$ such that $1 = h(t) = \|h\|_\infty$ and $h(t') = 0$. Let x denote the element of the c_0 -direct sum whose n -th coordinate is h if $n = N$ and 0 if $n \neq N$. Then $\lambda \|x^*x\| < N \|x^*x\| = N(1 + N) < (1 + N)^2 = \|x\|^2$, a contradiction.

3.3.8 Example. Consider $\mathcal{M}_2(\mathbb{C})$, the 2×2 matrices over the complex field, which is a B^* -algebra under some norm. If

$$a = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \text{ for any positive integer } n \text{ define}$$

$\|a\|_n = \max\{|a_{11}|, |a_{22}|\} + n(|a_{12}| + |a_{21}|)$. Then $a \mapsto \|a\|_n$ is a norm under which $\mathcal{M}_2(\mathcal{C})$ is a Banach $*$ -algebra. In fact, for a and b in $\mathcal{M}_2(\mathcal{C})$ and any complex number λ it is clear that $\|a+b\|_n \leq \|a\|_n + \|b\|_n$, $\|a\|_n = \|a^*\|_n$, and $\|\lambda a\|_n = |\lambda| \|a\|_n$. Furthermore, if a_{ij} and b_{ij} ($i, j=1, 2$) denote the entries of the matrices a and b as above, then

$$\begin{aligned}
 \|ab\|_n &= \max\{|a_{11}b_{11} + a_{12}b_{21}|, |a_{21}b_{12} + a_{22}b_{22}|\} \\
 &+ n(|a_{11}b_{12} + a_{12}b_{22}| + |a_{21}b_{11} + a_{22}b_{21}|) \\
 &\leq \max\{|a_{11}|, |a_{22}|\} \cdot \max\{|b_{11}|, |b_{22}|\} + |a_{12}| |b_{21}| + |a_{21}| |b_{12}| \\
 &+ n(|b_{12}| + |b_{21}|) \cdot \max\{|a_{11}|, |a_{22}|\} + n(|a_{12}| + |a_{21}|) \cdot \max\{|b_{11}|, |b_{22}|\} \\
 &\leq \|b\|_n \cdot \max\{|a_{11}|, |a_{22}|\} + |a_{12}|(n^2|b_{21}| + n \max\{|b_{11}|, |b_{22}|\}) + |a_{21}|(n^2|b_{12}| + n \max\{|b_{11}|, |b_{22}|\}) \\
 &\leq \|a\|_n \|b\|_n \quad \text{since } n \geq 1.
 \end{aligned}$$

Since all locally convex Hausdorff topologies on a finite dimensional space are equivalent [18, p. 37], the norms $a \mapsto \|a\|_n$ ($n=1, 2, \dots$) are equivalent to the B^* -norm. Hence, $c_0(\Sigma \oplus (\mathcal{M}_2(\mathcal{C}), a \mapsto \|a\|_n))$ is a non-commutative AB^* -algebra with bounded approximate identity by Theorem 3.3.4. Moreover, it is not B^* under any equivalent norm. If it were, by Lemma 3.3.6 there would exist a constant λ such that $\|x\|^2 \leq \lambda \|x^*x\|$ for each x in this algebra. Choose an integer $N > \lambda$. Define x to be the element of the c_0 -direct sum whose n -th coordinate is $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ if $n = N+1$ and 0 if $n \neq N+1$. Then $\lambda \|x^*x\| < N \|x^*x\| = N < (N+1)^2 = \|x\|^2$, a

contradiction.

Shirali and Ford [25] have shown that a Banach algebra with an involution $x \rightarrow x^*$ is symmetric if and only if the involution is hermetian. We are unable to settle the question whether an AB^* -algebra is symmetric or has an hermetian involution, but we give a partial result.

3.3.9 Theorem. If any commutative AB^* -algebra with identity is symmetric, then any AB^* -algebra with bounded approximate identity is symmetric.

Proof. Let A be an AB^* -algebra with bounded approximate identity. If A does not have an identity, form A_1 by adjoining an identity to A ; then we have that A_1 is AB^* by Theorem 3.3.5. If x is any element of A , then $Sp_{A_1}(x^*x) = Sp_A(x^*x)$ [16,p.32]. Let C denote A if A has an identity and A_1 if it does not.

There exists a closed maximal commutative $*$ -subalgebra B of C containing x^*x such that $Sp_B(x^*x) = sp_C(x^*x)$ [16, p. 182]. Moreover, B is an AB^* -algebra by Theorem 3.3.5. By hypothesis B is then symmetric; therefore, -1 is not an element of $Sp_B(x^*x)$ and so not in $Sp_A(x^*x)$. Thus, A is symmetric.

Both Examples 3.3.7 and 3.3.8 are symmetric. To see this, let T be a locally compact Hausdorff space, and let

$x = (x_n)$ be an element of $c_0(\Sigma\oplus(C_0(T), x_n \rightarrow \|x_n\|_n))$. For each n define $y_n(t) = 1 - 1/(|x_n(t)|^2 + 1)$ for t in T . Each y_n is an element of $C_0(T)$, and $\|y_n\|_n \leq (\|x_n\|_\infty)^2 + 2\|x_n\|_\infty \cdot \|x_n\|_n \leq 3(\|x_n\|_n)^2$. Therefore, $y = (y_n)$ is an element of $c_0(\Sigma\oplus(C_0(T), x_n \rightarrow \|x_n\|_n))$ and $yx^*x - x^*x + y = 0$. Thus, this c_0 -direct sum is symmetric.

If $a = (a_{ij})$ is an element of $\mathcal{M}_2(\mathcal{C})$, then a^*a is of the form (b_{ij}) , where b_{11} and b_{22} are non-negative, $b_{12} = \overline{b_{21}}$, and $b_{11}b_{22} - b_{21}b_{12} \geq 0$. If e denotes the identity of $\mathcal{M}_2(\mathcal{C})$, set $\delta_a = \det(a^*a + e)$. Then $\delta_a \geq 1$ and $(1/\delta_a) \begin{pmatrix} b_{22} + 1 & -b_{12} \\ -b_{21} & b_{11} + 1 \end{pmatrix}$ is the inverse of $a^*a + e$ in $\mathcal{M}_2(\mathcal{C})$. Moreover, for each positive integer n we have that $\|e - (a^*a + e)^{-1}\|_n \leq (1/\delta_a) \|a^*a\|_n (\|a^*a\|_n + 1) \leq (\|a\|_n)^4 + (\|a\|_n)^2$ since $\delta_a \geq 1$.

Consequently, if (a_n) is an element of $c_0(\Sigma\oplus(\mathcal{M}_2(\mathcal{C}), x_n \rightarrow \|x_n\|_n))$, then setting $b_n = e - (a_n^*a_n + e)^{-1}$, we see that (b_n) is an element of this c_0 -direct sum and that $(a_n^*a_n)(b_n) - (a_n^*a_n) + (b_n) = 0 = (b_n)(a_n^*a_n) - (a_n^*a_n) + (b_n)$. Hence, $c_0(\Sigma\oplus(\mathcal{M}_2(\mathcal{C}), x_n \rightarrow \|x_n\|_n))$ is symmetric.

In section 2 we showed that if A is an AB^* -algebra with bounded approximate identity, then the Arens multiplications are regular on A^{**} . The converse of this theorem is not true.

3.3.10 Example. Let A be the set of all complex-valued functions defined and continuous on the closed unit disc D of the complex plane whose restrictions to the interior of D are holomorphic. For x in A define $x^*(\alpha) = \overline{x(\overline{\alpha})}$ for all α in D . Then $x \rightarrow x^*$ is an involution on A , and we note in passing that it is not hermitian. With respect to this involution A is a commutative Banach $*$ -algebra with identity. The Arens multiplications are regular on A^{**} [4]; however, A is not an AB^* -algebra.

The example is a variation of a proof of Naimark [15,p.274], in which he shows that any linear functional of A^* with the property $f(x) = \overline{f(x^*)}$ for all x in A is not the difference of two linear functionals f_1 and f_2 in A^* having the property that $f_i(x^*x) \geq 0$ for all x in A .

Let x_0 be any element of A such that $\sup\{|x_0(\alpha)| : -1 \leq \alpha \leq 1\} < 1$ and $\|x_0\|_\infty \geq 1$; for example, let $x_0(\alpha) = i \sin(\alpha)$. Choose α_0 in D such that $\|x_0\|_\infty = |x_0(\alpha_0)|$. Define a linear functional on A by $g(x) = x(\alpha_0)$; then g is an element of A^* . Setting P equal to the set $\{\theta \in A^* : \theta(x^*x) \geq 0, \forall x \in A\}$, we show that g is not an element of the norm-closure of $\langle P \rangle$ and, thus, that A is not AB^* in view of Theorem 3.2.14. Suppose g were an element of this set: then there would exist θ_i in P and complex numbers λ_i ($i=1,2,3,4$) such that $\|g - \sum_{i=1}^4 \lambda_i \theta_i\| < \frac{1}{2}$.

If Φ is the carrier space of A , we denote by Φ^* the set of fixed points of the homeomorphism $\varphi \rightarrow \varphi^*$ on Φ defined by $\varphi^*(x) = \overline{\varphi(x^*)}$ for x in A . The set Φ^* is non-empty and is in one-one correspondence with the points in the closed interval $[-1,1]$. In fact, for each φ in Φ^* there is a point α in $[-1,1]$ such that $\varphi(x) = x(\alpha)$ for all x in A [16,p.304]. Letting $x \rightarrow \hat{x}$ be the Gelfand transform on A , we have by Theorem 4.6.15 of [16] that each $\theta_i (i=1,2,3,4)$ is of the form $\theta_i(x) = \int_{\Phi^*} \hat{x}(\varphi) dm_i(\varphi)$ for x in A , where m_i is a regular non-negative real measure defined on the Borel subsets of Φ^* .

By the identification of the set Φ^* with the closed interval $[-1,1]$ we see that $\sup\{|\hat{x}(\varphi)| : \varphi \in \Phi^*\} = \sup\{|x(\alpha)| : -1 \leq \alpha \leq 1\}$ for each x in A .

Let $\|m_i\| (i=1,2,3,4)$ denote the total variation of m_i . Since $\sup\{|x_0(\alpha)| : -1 \leq \alpha \leq 1\} < 1$, there is a positive integer n such that $\sup\{|x_0^n(\alpha)| : -1 \leq \alpha \leq 1\} < 1/(2\sum_{i=1}^4 |\lambda_i| \|m_i\| + 2)$. Furthermore, $\|x_0^n\|_\infty = |x_0(\alpha_0)|^n \geq 1$. Let $y(\alpha) = (1/|x_0^n(\alpha_0)|)x_0^n(\alpha)$ for α in D . Then y is an element of A , $\|y\|_\infty = |y(\alpha_0)| = 1$, and $\sup\{|y(\alpha)| : -1 \leq \alpha \leq 1\} \leq 1/(2\sum_{i=1}^4 |\lambda_i| \|m_i\| + 2)$. Therefore, $|g(y) - \sum_{i=1}^4 \lambda_i \theta_i(y)| < \frac{1}{2}$ in view of the manner in which θ_i and λ_i were chosen. Hence, from the definition of g and the integral representations for θ_i we have that $1 = |y(\alpha_0)| = |g(y)| < \frac{1}{2} + |\sum_{i=1}^4 \lambda_i \theta_i(y)| \leq \frac{1}{2} + \sum_{i=1}^4 |\int_{\Phi^*} \lambda_i \hat{y}(\varphi) dm_i(\varphi)|$

$$\leq \frac{1}{2} + \sum_{i=1}^4 |\lambda_i| \|m_i\| \cdot \sup\{|\hat{y}(\varphi)| : \varphi \in \mathfrak{F}^*\}$$

$$= \frac{1}{2} + \sum_{i=1}^4 |\lambda_i| \|m_i\| \cdot \sup\{|y(\alpha)| : -1 \leq \alpha \leq 1\} \leq \frac{1}{2} + \frac{1}{2} = 1, \text{ a}$$

contradiction. Therefore, g is an element of A^* which is not in the norm closure of $\langle P \rangle$, as was to be shown.

Finally, there are many Banach $*$ -algebras for which the Arens multiplications are not regular on the bidual.

3.3.11 Example. Let G be an infinite locally compact abelian group. Then the Arens multiplications on the bidual of the commutative Banach $*$ -algebra $L^1(G)$ are never regular [4].

CHAPTER IV

ARENS MULTIPLICATIONS ON LOCALLY CONVEX COMPLETIONS OF BANACH ALGEBRAS

If (E, \mathfrak{T}) is a locally convex Hausdorff space, then there exists a complete locally convex Hausdorff space E^\wedge in which E can be densely imbedded; and the space E^\wedge may be identified with a certain subspace of $(E, \mathfrak{T})^*$, the set of all linear functionals on $(E, \mathfrak{T})^*$ (see Theorem 4.1.1). We are interested in what properties on \mathfrak{T} ensure us that an Arens multiplication can be defined on E^\wedge with respect to E and $(E, \mathfrak{T})^*$ (see Definition 3.1.1); and if this is the case, whether or not E^\wedge is an algebra under this multiplication.

In section 1 we give sufficient conditions for either of the Arens multiplications to be defined on E^\wedge and for E^\wedge to be an algebra under that multiplication. We also provide an example of a locally convex completion on which neither Arens multiplication can be defined. We do not know if a locally convex completion must be an algebra under an Arens multiplication which can be defined on it; and we investigate this problem in subsequent sections from the viewpoint of specific examples: the completions of a Banach algebra with respect to the topologies in the β , μ , and ν

families.

Section 1. Sufficient Conditions that a Locally Convex Completion be an Algebra.

4.1.1 (Grothendieck Completion) Theorem. Let (E, \mathfrak{J}) be a locally convex Hausdorff space, and let \mathcal{U} be a base of neighborhoods of zero. Denote by $(E, \mathfrak{J})^\wedge$ the set of all linear functionals on $(E, \mathfrak{J})^*$ which are $\sigma((E, \mathfrak{J})^*, E)$ -continuous on the polar of each U in \mathcal{U} . Then $(E, \mathfrak{J})^\wedge$ endowed with the topology whose neighborhoods are all subsets of $(E, \mathfrak{J})^\wedge$ containing a set of the form $U^{oo} \cap (E, \mathfrak{J})^\wedge$, where U is a \mathcal{U} and the bipolar is taken in $(E, \mathfrak{J})^{*'} ,$ is a complete locally convex Hausdorff space in which E is imbedded as a dense subspace by the natural injection. Moreover, any complete locally convex Hausdorff space in which E can be imbedded as a dense subspace is (linearly) isomorphic to $(E, \mathfrak{J})^\wedge$.

Proof. The reader is referred to [18, pp.106-8].

4.1.2 Definition. The set $(E, \mathfrak{J})^\wedge$ with the topology described in Theorem 4.1.1 is called the completion of E with respect to \mathfrak{J} and the aforementioned topology on $(E, \mathfrak{J})^\wedge$ is called the \mathfrak{J} -completion topology. If the topology \mathfrak{J} need not be explicitly mentioned, we simply say that E^\wedge is a locally convex completion.

4.1.3 Corollary. With the notation as in the theorem let \mathcal{V}

be a subbase of \mathfrak{J} -neighborhoods of zero consisting of closed absolutely convex sets V such that λV is in \mathcal{V} and every non-zero complex number λ . Then $(E, \mathfrak{J})^\wedge$ is the set of all linear functionals on $(E, \mathfrak{J})^*$ which are $\sigma((E, \mathfrak{J})^*, E)$ -continuous on the polar of each V in \mathcal{V} , and all the sets $V^{\circ\circ} \cap (E, \mathfrak{J})^\wedge$ form a subbase for the completion topology on $(E, \mathfrak{J})^\wedge$.

Proof. Since \mathcal{V} is a subbase of \mathfrak{J} -neighborhoods, the collection of all finite intersections of elements of \mathcal{V} form a base for \mathfrak{J} -neighborhoods; let us call it \mathcal{U} . Then clearly any element of $(E, \mathfrak{J})^\wedge$ is $\sigma((E, \mathfrak{J})^*, E)$ -continuous on the polar of each V in \mathcal{V} .

Conversely, assume that F is a linear functional on $(E, \mathfrak{J})^*$ which is $\sigma((E, \mathfrak{J})^*, E)$ -continuous on the polar of each V in \mathcal{V} . We show that F is $\sigma((E, \mathfrak{J})^*, E)$ -continuous on the polar of an element U in \mathcal{U} of the form $U = V_1 \cap V_2$. The general case then follows by induction. Since V_1 and V_2 are absolutely convex and $\sigma(E, \mathfrak{J})^*$ -closed [18, p.34], the set U° in $(E, \mathfrak{J})^*$ is the $\sigma((E, \mathfrak{J})^*, E)$ -closed absolutely convex hull of $V_1^\circ \cup V_2^\circ$ [18, p.36]. Moreover, V_1° and V_2° are $\sigma((E, \mathfrak{J})^*, E)$ -compact [18, p.61]; therefore, $V_1^\circ + V_2^\circ$ is $\sigma((E, \mathfrak{J})^*, E)$ -closed, absolutely convex, and contains $V_1^\circ \cup V_2^\circ$. Hence, U° is contained in $V_1^\circ + V_2^\circ$.

To show that F is $\sigma((E, \mathfrak{J})^*, E)$ -continuous on U° , it suffices to show that F is $\sigma((E, \mathfrak{J})^*, E)$ -continuous on U° .

at the point 0 [18,p.102]. Let $\{f_\alpha\}$ be a net in U^0 converging to 0 in the $\sigma((E,\mathcal{T})^*,E)$ topology. Each f_α can be written as $f_\alpha = f_{\alpha,1} + f_{\alpha,2}$, where $f_{\alpha,i}$ is in V_i^0 ($i=1,2$). Since V_1^0 is $\sigma((E,\mathcal{T})^*,E)$ -compact, there is an element g_1 in V_1^0 which is a $\sigma((E,\mathcal{T})^*,E)$ -cluster point of the net $\{f_{\alpha,1}\}$. Choose a subnet $\{f_{\beta,1}\}$ of $\{f_{\alpha,1}\}$ for which g_1 is its $\sigma((E,\mathcal{T})^*,E)$ -limit. Then $\{f_{\beta,2}\}$ is a subnet of $\{f_{\alpha,2}\}$ in the $\sigma((E,\mathcal{T})^*,E)$ -compact set V_2^0 . Let g_2 be a $\sigma((E,\mathcal{T})^*,E)$ -cluster point of the net $\{f_{\beta,2}\}$. Then $g_1 + g_2$ is a $\sigma((E,\mathcal{T})^*,E)$ -cluster point of the net $\{f_{\beta,1} + f_{\beta,2}\}$, a subnet of the original net $\{f_\alpha\}$ which converges to 0 in the $\sigma((E,\mathcal{T})^*,E)$ topology. Therefore, $g_1 + g_2 = 0$.

By the hypothesis on F we have that $F(g_1) = \lim F(f_{\beta,1})$ and that $F(g_2)$ is a cluster point of the net $\{F(f_{\beta,2})\}$. Therefore, $0 = F(0) = F(g_1 + g_2) = F(g_1) + F(g_2)$ is a cluster point of the net $\{F(f_{\beta,1}) + F(f_{\beta,2})\}$, which is the same as the net $\{F(f_\alpha)\}$. It now follows that F is $\sigma((E,\mathcal{T})^*,E)$ -continuous on U^0 at the point 0 .

To show that the sets $V^{00} \cap (E,\mathcal{T})^\wedge$ form a subbase for the completion topology on $(E,\mathcal{T})^\wedge$, it suffices to show that any set of the form $(V_1 \cap \dots \cap V_n)^{00} \cap (E,\mathcal{T})^\wedge$, where each V_i is in \mathcal{V} , contains a set of the form $W_1^{00} \cap \dots \cap W_m^{00} \cap (E,\mathcal{T})^\wedge$, where each W_i is in \mathcal{V} . We show that for two elements V_1 and V_2 in \mathcal{V} that $(\frac{1}{2}V_1)^{00} \cap (\frac{1}{2}V_2)^{00} \cap (E,\mathcal{T})^\wedge$ is contained

in $(V_1 \cap V_2)^{\circ\circ} \cap (E, \mathfrak{T})^\wedge$ and note that both $\frac{1}{2}V_1$ and $\frac{1}{2}V_2$ are elements of \mathcal{V} . The general case follows by an induction argument.

Let F be an element of $(\frac{1}{2}V_1)^{\circ\circ} \cap (E, \mathfrak{T})^\wedge$
 $= \frac{1}{2}V_1^{\circ\circ} \cap \frac{1}{2}V_2^{\circ\circ} \cap (E, \mathfrak{T})^\wedge$ and f be an element of $(V_1 \cap V_2)^\circ$.
 As we noted before, $(V_1 \cap V_2)^\circ$ is contained in $V_1^\circ + V_2^\circ$;
 therefore, there are elements g_1 in V_1° and g_2 in V_2°
 such that $f = g_1 + g_2$. Hence, $|F(f)| \leq |F(g_1)| + |F(g_2)| \leq \frac{1}{2} + \frac{1}{2}$;
 so that F is an element of $(V_1 \cap V_2)^{\circ\circ} \cap (E, \mathfrak{T})^\wedge$. The proof
 is now concluded.

Let A be a Banach algebra and \mathfrak{T} be a locally convex Hausdorff topology on A such that $\mathfrak{T} \leq \tau(A, A^*)$; in particular, $(A, \mathfrak{T})^*$ is a subspace of A^* . Using the notation established in Definition 3.1.1 and that A^* is both a left and a right A -module, we see that the Arens multiplication $(F, G) \rightarrow FG$ (resp., $F:G$) can be defined on $(A, \mathfrak{T})^\wedge$ with respect to A and $(A, \mathfrak{T})^*$ if $(A, \mathfrak{T})^*$ is a right (resp., left) A -module and if $(A, \mathfrak{T})^*$ is invariant under the mapping $(F, f) \rightarrow F \cdot f$ (resp., $F:f$) for F in $(A, \mathfrak{T})^\wedge$ and f in $(A, \mathfrak{T})^*$.

4.1.4. Lemma. Let A be a Banach algebra and \mathfrak{T} be a locally convex Hausdorff topology on A such that $\mathfrak{T} \leq \tau(A, A^*)$. Then if the mapping $x \rightarrow ax$ (resp., xa) is \mathfrak{T} -continuous on A for each a in A , then $(A, \mathfrak{T})^*$ is an

isometric right (resp., left) A -module.

Proof. Let a be an element of A and f be an element of $(A, \mathfrak{J})^*$. Then there is a \mathfrak{J} -neighborhood U such that f is an element of U^0 in $(A, \mathfrak{J})^*$. If the mapping $x \rightarrow ax$ is \mathfrak{J} -continuous, there exists a \mathfrak{J} -neighborhood V such that $a \cdot V$ is contained in U . Therefore, $f \cdot a$ is an element of $U^0 \cdot a$ and this set is contained in V^0 . Hence, $f \cdot a$ is an element of $(A, \mathfrak{J})^*$ and $\|f \cdot a\| \leq \|f\| \|a\|$. Moreover, it is easy to see that the mapping $(a, f) \rightarrow f \cdot a$ from $A \times (A, \mathfrak{J})^*$ to $(A, \mathfrak{J})^*$ is a modular action.

4.1.5 Theorem. Let A be a Banach algebra and \mathfrak{J} be a locally convex Hausdorff topology on A such that $\mathfrak{J} \leq \tau(A, A^*)$ and such that the following properties hold: (i) the mapping $x \rightarrow ax$ (resp., xa) is \mathfrak{J} -continuous on A for each a in A ; (ii) the space $(A, \mathfrak{J})^*$ is invariant under the mapping $(F, f) \rightarrow F \cdot f$ (resp., $F : f$) for F in $(A, \mathfrak{J})^\wedge$ and f in $(A, \mathfrak{J})^*$; and (iii) for each F in $(A, \mathfrak{J})^\wedge$ and each \mathfrak{J} -neighborhood U , there exists a \mathfrak{J} -neighborhood V such that $G \cdot U^0$ (resp., $G : U^0$) is contained in V^0 . Then $(A, \mathfrak{J})^\wedge$ is an algebra under the Arens multiplication $(F, G) \rightarrow FG$ (resp., $F : G$). Furthermore, the mapping $F \rightarrow FG$ (resp., $F : G$) on $(A, \mathfrak{J})^\wedge$ for each G in $(A, \mathfrak{J})^\wedge$ is continuous in the completion topology.

Proof. We prove the theorem for the Arens multipli-

cation $(F, G) \rightarrow FG$; the proof for the other multiplication is analogous. Lemma 4.1.4 and property (ii) ensures us that this Arens multiplication can be defined on $(A, \mathfrak{I})^\wedge$. Then FG is in $(A, \mathfrak{I})^*$ and is in $(A, \mathfrak{I})^\wedge$ as well if it is $\sigma((A, \mathfrak{I})^*, A)$ -continuous on the polar of each \mathfrak{I} -neighborhood.

Let U be a \mathfrak{I} -neighborhood, and let $\{f_\alpha\}$ be a net in U° converging to 0 in the $\sigma((A, \mathfrak{I})^*, A)$ topology. For each a in A the net $\{f_\alpha \cdot a\}$ converges to 0 in the $\sigma((A, \mathfrak{I})^*, A)$ topology. Since the mapping $x \rightarrow ax$ is \mathfrak{I} -continuous, there exists a \mathfrak{I} -neighborhood V such that $a \cdot V$ is contained in U . Then $U^\circ \cdot a$ is contained in V° ; hence, the set $\{f_\alpha \cdot a\}$ is also contained in V° . Since G is an element of $(A, \mathfrak{I})^\wedge$, we have that $\lim G \cdot f_\alpha(a) = \lim G(f_\alpha \cdot a) = 0$ for each a in A . Moreover, there is a \mathfrak{I} -neighborhood W such that $G \cdot U^\circ$ is contained in W° ; so $\lim FG(F_\alpha) = \lim F(G \cdot f_\alpha) = 0$ since F is an element of $(A, \mathfrak{I})^\wedge$. Therefore, FG is an element of $(A, \mathfrak{I})^\wedge$ for every F and G in $(A, \mathfrak{I})^\wedge$. That $(A, \mathfrak{I})^\wedge$ is an algebra follows from Lemma 3.1.2 and the remark that the mapping $(F, G) \rightarrow FG$ is bilinear.

Finally, let G be any element of $(A, \mathfrak{I})^\wedge$ and let $\{F_\alpha\}$ be a net in $(A, \mathfrak{I})^\wedge$ which converges to 0 in the completion topology. If U is a \mathfrak{I} -neighborhood, there exists a \mathfrak{I} -neighborhood V such that $G \cdot U^\circ$ is contained in V° in $(A, \mathfrak{I})^*$. Furthermore, there is an α_0 such that for $\alpha > \alpha_0$ we have F_α in $V^{\circ\circ} \cap (A, \mathfrak{I})^\wedge$, a subset of

$(G \cdot U^0)^0 \cap (A, \mathfrak{I})^0$ in $(A, \mathfrak{I})^*$. Therefore, $F_\alpha G$ is an element of $U^{00} \cap (A, \mathfrak{I})^\wedge$ for each $\alpha > \alpha_0$. Hence, the net $\{F_\alpha G\}$ converges to 0 in the completion topology.

In Theorem 4.1.5 we note that properties (i) and (iii) imply that multiplication is separately \mathfrak{I} -continuous on A . However, separate \mathfrak{I} -continuity of multiplication does not imply property (ii).

4.1.6 Example. Let S be the closed unit interval $[0,1]$ with the usual topology; and consider $C(S)$, the continuous complex-valued functions on S . Under pointwise operations and norm defined by $\|f\| = \sup \{|f(s)| : s \in S\}$, $C(S)$ is a Banach algebra. Then multiplication on $C(S)$ is separately continuous in the $\sigma(C(S), C(S)^*)$ topology; however, the topological dual of $C(S)$ with respect to this topology, which is the same as the norm dual $C(S)^*$, is not invariant under the mapping $(F, f) \rightarrow F \cdot f$ for F in $(C(S), \sigma(C(S), C(S)^*)^\wedge$ and f in $C(S)^*$.

For each positive integer n define

$$x_n(s) = \begin{cases} s - (2^{n-1} - 1)/2^{n-1} & \text{for } (2^{n-1} - 1)/2^{n-1} \leq s \leq (2^{n+1} - 3)/2^{n+1} \\ (2^n - 1)/2^n - s & \text{for } (2^{n+1} - 3)/2^{n+1} \leq s \leq (2^n - 1)/2^n \\ 0 & \text{elsewhere} \end{cases}$$

Then each x_n is an element of $C(S)$ and $\|x_n\| = 1/2^{n+1}$. Let μ be Lebesgue measure on A ; then it

is easily shown that the set $\{\mu \cdot x_n\}$ is linearly independent. Consequently, there is an element F in $C(S)^*$, which is the completion of $C(S)$ with respect to the $\sigma(C(S), C(S)^*)$ topology, such that $F(\mu \cdot x_n) = n$, $n = 1, 2, \dots$; in particular, $F \cdot \mu$ is not an element of $C(S)^*$.

Section 2. The β -Completions of a Banach Algebra with Bounded Approximate Identity.

The completions with respect to β^1 and β^2 of a Banach algebra A with a bounded approximate identity are algebras under certain Arens multiplications and are isomorphic to the algebras of left and right centralizers on A respectively. Both Arens multiplications can be defined on the β -completion, but we prove it is an algebra under either multiplication only with the assumption that $A^* = (A, \beta)^*$; and, in this case, $(A, \beta)^\wedge$ is isomorphic (under either Arens multiplication) to the algebra of double centralizers on A . Under certain conditions each of the β completions can be realized as subsets of A^{**} .

If A is a B^* -algebra, then $(A, \beta)^\wedge$ (under either multiplication) is a B^* -subalgebra of A^{**} and is, in fact, the maximal subalgebra of A^{**} containing A as a two-sided ideal. Moreover, we prove that $(A, \beta)^\wedge$ is exactly A^{**} if and only if A is dual: that is, if and only if A is isomorphic to a B^* -subalgebra of the compact operators

on some Hilbert space.

The concepts in the next definition and proofs of the two subsequent theorems are given by B. E. Johnson in [13] and [14].

4.2.1 Definition. If A is a Banach algebra, let $L(A)$ (resp., $R(A)$) be the set of all operators T on A such that $T(ab) = (Ta)b$ (resp., $T(ab) = a(Tb)$) for all a and b in A . The set $L(A)$ (resp., $R(A)$) is called the set of left (resp., right) centralizers on A . Let $M(A)$ be the set of all pairs (S, S') where S and S' are operators on A satisfying the relation $a(Sb) = (S'a)b$ for all a and b in A ; the set $M(A)$ is called the set of double centralizers on A .

4.2.2 Theorem. If A is a Banach algebra with bounded approximate identity, then (i) each left (right) centralizer on A is linear and continuous; and (ii) under the operations defined by $(T+T')a = Ta + T'a$, $(\lambda T)a = \lambda(Ta)$, and $(TT')a = T(T'a)$, where a is in A and λ is a complex number; and with norm defined by $\|T\| = \sup\{\|Tx\| : x \in A, \|x\| \leq 1\}$, the set $L(A)$ (resp., $R(A)$) is a Banach algebra with identity. Furthermore, as an algebra A is imbedded isometrically in $L(A)$ by the mapping $a \rightarrow L_a$, where $L_ax = ax$ (resp., as an algebra A is imbedded anti-isomorphically in $R(A)$ by the mapping $a \rightarrow R_a$, where $R_ax = xa$) for x in A .

4.2.3 Theorem. If A is a Banach algebra with bounded approximate identity, then (i) for each (S, S') in $M(A)$, the operator S is in $L(A)$ and S' is in $R(A)$; in particular, S and S' are continuous linear operators on A such that $\|S\| = \|S'\|$; and (ii) under the operations defined by $(S, S') + (T, T') = (S+T, S'+T')$, $\lambda(S, S') = (\lambda S, \lambda S')$ for any complex number λ , and $(S, S')(T, T') = (ST, T'S')$; and with norm defined by $\|(S, S')\| = \|S\|$, the set $M(A)$ is a Banach algebra with identity. Furthermore, as an algebra A is imbedded isometrically in $M(A)$ by the mapping $a \mapsto (L_a, R_a)$, where $L_a x = ax$ and $R_a x = xa$ for x in A .

For the remainder of the section we assume that A denotes a Banach algebra with bounded approximate identity; thus, the topologies β^1, β^2 , and β are Hausdorff on A . We now prove that $(A, \beta^1)^\wedge$ and $(A, \beta^2)^\wedge$ are algebras under certain Arens multiplications and are isomorphic to $R(A)$ and $L(A)$ respectively.

4.2.4 Lemma. If a is an element of A and $U = \{x \in A : \|ax\| \leq 1\}$, then $U^0 = \{f \cdot a : f \in A^*, \|f\| \leq 1\}$ in A^* . If $V = \{x \in A : \|xa\| \leq 1\}$, then $V^0 = \{a \cdot f : f \in A^*, \|f\| \leq 1\}$ in A^* .

Proof. Let g be an element of U^0 . Define a linear functional f on the subspace $a \cdot A = \{ax : x \in A\}$ by $f(ax) = g(x)$. Since $|g(x)| \leq \|ax\|$ for all x in A by [18, p.13], the linear functional f is well-defined and bounded by 1 on $a \cdot A$;

therefore, it can be extended to a continuous linear functional \bar{f} on A such that $\|\bar{f}\| \leq 1$ and $\bar{f} \cdot a = g$. Thus, U^0 is contained in the set $\{f \cdot a : f \in A^*, \|f\| \leq 1\}$, and the reverse containment is clear. The proof for V^0 is similar.

The results stated in the next lemma and subsequent theorems for the β^1 and β^2 topologies have similar proofs; however, the former case involves more relations on the Arens multiplications (see Lemma 3.1.2). Therefore, we always prove the results for β^1 .

In accord with the convention established in section 1 of Chapter III we continue to regard each element of A as an element of $(A, \beta^1)^\wedge$ and, in particular, as a linear functional on $(A, \beta^1)^*$. However, limitations in our notation require that special care must be taken as, for instance, in the next lemma.

4.2.5 Lemma. For each F in $(A, \beta^1)^\wedge$ (resp., $(A, \beta^2)^\wedge$) and for each a in A there exists a unique element b in A such that $F(f \cdot a) = f(b)$ (resp., $F(a \cdot f) = f(b)$) for all f in A^* .

Proof. We have that $(A, \beta^1)^* = A^* \cdot A$ by Theorem 2.1.7; and since F is a linear functional on $(A, \beta^1)^*$, we see that $F(f \cdot a)$ is well-defined for each f in A^* (note that $F : a$ is defined as a linear functional only on $(A, \beta^1)^*$).

Let $U_n = \{x \in A : \|ax\| \leq n\}$ for each positive integer n . Since A is dense in $(A, \beta^1)^\wedge$ with respect to the completion topology and each $U_n^{\circ\circ} \cap (A, \beta^1)^\wedge$ in $(A, \beta^1)^*$ is a completion-topology neighborhood by Theorem 4.1.1, there exists a sequence $\{c_n\}$ in A such that $F \cdot c_n$ is an element of $U_n^{\circ\circ} \cap (A, \beta^1)^\wedge$ for each n . If S_0 denotes the unit ball of A^* , then $U_n^0 = (1/n)S_0 \cdot a$ by Lemma 4.2.4. Therefore $|F(f \cdot a) - f \cdot a(c_n)| \leq 1/n$ for each f in S_0 ; hence, the sequence $\{ac_n\}$ is norm-Cauchy in the Banach algebra A . Let $b = \lim ac_n$ in A . Then for each f in A^* we have that $f(b) = \lim f(ac_n) = \lim f \cdot a(c_n) = F(f \cdot a)$ and that b is unique with respect to this property since A^* separates points of A .

4.2.6 Theorem. The completion of A with respect to β^1 (resp., β^2) is an algebra under the Arens multiplication $(F, G) \rightarrow F:G$ (resp., FG).

Proof. We show that $(A, \beta^1)^\wedge$ is an algebra by verifying the hypotheses of Theorem 4.1.5. First, multiplication on A is separately β^1 -continuous by Theorem 2.1.4. Next, we must show that $(A, \beta^1)^*$ is invariant under the mapping $(F, f) \rightarrow F:f$ for F in $(A, \beta^1)^\wedge$ and f in $(A, \beta^1)^*$. For any f in $(A, \beta^1)^*$ there are elements a in A and g in A^* such that $f = g \cdot a$ by Theorem 2.1.7. If F is any element of $(A, \beta^1)^\wedge$, by Lemma 4.2.5 there is an element

b in A such that $F(h \cdot a) = h(b)$ for every h in A^* . Then $F:f$ is an element of A' , and for each x in A we have $F:f(x) = F(x \cdot f) = F((x \cdot g) \cdot a) = x \cdot g(b) = g \cdot b(x)$. Therefore, $F:f = g \cdot b$ is an element of $(A, \beta^1)^*$ by Theorem 2.1.7.

The remaining hypothesis of Theorem 4.1.5 to be satisfied is that for each F in $(A, \beta^1)^\wedge$ and each β^1 -neighborhood U there is a β^1 -neighborhood V such that $F:U^0$ is contained in V^0 . Any β^1 -neighborhood U contains a set W of the form $\{x \in A: \|ax\| \leq 1\}$ by the remarks following Definition 2.1.1. For F in $(A, \beta^1)^\wedge$ choose b by Lemma 4.2.5 such that $F(f \cdot a) = f(b)$ for every f in A^* . Then, if S_0 denotes the unit ball of A^* , we have that $F:U^0$ is contained in $F:W^0 = F(S_0 \cdot a)$ by Lemma 4.2.4; moreover, for any g in S_0 and x in A we also have that $F((x \cdot g) \cdot a) = x \cdot g(b) = g \cdot b(x)$. Therefore, the subset $F:U^0$ of $F:(S_0 \cdot a)$ is contained in $S_0 \cdot b$, the polar of the β^1 -neighborhood $\{x \in A: \|bx\| \leq 1\}$ by Lemma 4.2.4. That $(A, \beta^1)^\wedge$ is an algebra now follows from Theorem 4.1.5.

4.2.7 Theorem. The algebra $(A, \beta^1)^\wedge$ (resp., $(A, \beta^2)^\wedge$) is isomorphic to $R(A)$ (resp., $L(A)$); moreover, there exists only one isomorphism $\Gamma: (A, \beta^1)^\wedge \rightarrow R(A)$ (resp., $\Gamma: (A, \beta^2)^\wedge \rightarrow L(A)$) such that $(\Gamma a)x = xa = R_a x$ (resp., $(\Gamma a)x = ax = L_a x$) for each a and x in A .

Proof. To each F in $(A, \beta^1)^\wedge$ and each a in A there exists a unique element $\gamma(F, a)$ in A such that $F(f \cdot a) = f(\gamma(F, a))$ for all f in A^* by Lemma 4.2.5. Therefore, the mapping $(F, a) \rightarrow \gamma(F, a)$ is easily shown to be bilinear. If F is an element of $(A, \beta^1)^\wedge$ and a and b are elements of A , then $f(\gamma(F, ab)) = F(f \cdot ab) = F((f \cdot a) \cdot b) = f \cdot a(\gamma(F, b)) = f(a \cdot \gamma(F, b))$ for each f in A^* ; thus $\gamma(F, ab) = a \cdot \gamma(F, b)$.

Define $\Gamma: (A, \beta^1) \rightarrow R(A)$ by $(\Gamma F)a = \gamma(F, a)$ for each a in A . The remarks above ensure us that Γ is well-defined and linear.

We now prove that $\Gamma(F:G) = (\Gamma F)(\Gamma G)$ for F and G in $(A, \beta^1)^\wedge$. Note that for f in $(A, \beta^1)^*$ and for a and x in A we have that $(G:a):f(x) = (G:a)(x \cdot f) = G(a:(x \cdot f)) = G((x \cdot f) \cdot a)$, where $(x \cdot f) \cdot a$ is a well-defined element of $(A, \beta^1)^*$ by Theorem 2.1.7. Hence, $(G:a):f(x) = x \cdot f(\gamma(G, a)) = f \cdot \gamma(G, a)(x)$; in particular, $(G:a):f = f \cdot \gamma(G, a)$. Now, for F and G in $(A, \beta^1)^\wedge$, a in A , and f in $(A, \beta^1)^*$ we have that $f(\Gamma(F:G)a) = f(\gamma(F:G, a)) = F:G(f \cdot a) = F:G(a:f) = (F:G):a(f) = F:(G:a)(f) = F((G:a):f) = F(f \cdot \gamma(G, a)) = f(\gamma(F, \gamma(G, a)))$. Since $(A, \beta^1)^*$ separates points of A , we see that $\Gamma(F:G)a = \gamma(F, \gamma(G, a)) = (\Gamma F)(\gamma(G, a)) = (\Gamma F)((\Gamma G)a) = (\Gamma F)(\Gamma G)a$. Consequently, $\Gamma(F:G) = (\Gamma F)(\Gamma G)$ for every F and G in $(A, \beta^1)^\wedge$.

To show that the mapping Γ is injective, let F and

G be unequal elements of $(A, \beta^1)^\wedge$. Then there exists an f in $(A, \beta^1)^*$ such that $F(f) \neq G(f)$ and $f = g \cdot a$ for some a in A and g in A^* by Theorem 2.1.7. Hence, $g((\Gamma F)a) = g(\gamma(F, a)) = F(g \cdot a) = F(f) \neq G(f) = G(g \cdot a) = g(\gamma(G, a)) = g((\Gamma G)a)$; therefore, $\Gamma F \neq \Gamma G$.

We now show that Γ is surjective. Let S be a non-zero element of $R(A)$, and let $\{e_\alpha\}$ be a bounded approximate identity for A . Then the set $\{Se_\alpha\}$ is a β^1 -Cauchy net in A : since S is bounded by Theorem 4.2.2, we have for each a in A and $\epsilon > 0$ that there is an α_0 so that $\|ae_\alpha - ae_\beta\| < \epsilon/\|S\|$ whenever $\alpha, \beta > \alpha_0$; consequently, $\|a(Se_\alpha - Se_\beta)\| = \|S(ae_\alpha - ae_\beta)\| < \epsilon$ for $\alpha, \beta > \alpha_0$. There is an element F in $(A, \beta^1)^\wedge$ such that F is the limit of the net $\{Se_\alpha\}$ in the completion topology by Theorem 4.1.1. It remains to show that $\Gamma F = S$.

For each a in A and $\epsilon > 0$ there is an α_0 such that for $\alpha > \alpha_0$ we have that $\|ae_\alpha - a\| < \epsilon/2 \cdot \|S\|$ and that $F - Se_\alpha$ is an element of the set $\{x \in A: \|ax\| \leq 2/\epsilon\}^\infty = (\epsilon/2) \cdot \{f \cdot a \in (A, \beta^1)^*: f \in A^*, \|f\| \leq 1\}^0$ in $(A, \beta^1)^{*1}$ by Lemma 4.2.4. Therefore, for each $\alpha > \alpha_0$ and each f in A^* of norm at most 1 we see that $|f((\Gamma) a) - f(S(ae_\alpha))| = |f(\gamma(F, a)) - f(a(Se_\alpha))| = |F(f \cdot a) - f \cdot a(Se_\alpha)| \leq \epsilon/2$ and so $|f((\Gamma F)a) - f(Sa)| \leq |f((\Gamma F)a) - f(S(ae_\alpha))| + |f(S(ae_\alpha)) - f(Sa)| \leq \epsilon/2 + \|S\| \|ae_\alpha - a\| < \epsilon$. Consequently, $(\Gamma F)a = Sa$ for every a in A , and the mapping Γ is surjective.

We have now shown that $\Gamma: (A, \beta^1)^\wedge \rightarrow R(A)$ is an algebra isomorphism. Moreover, $(\Gamma a)x = \gamma(a, x) = xa = R_a x$ by the definition of $\gamma(a, x)$ for every a and x in A . It remains to show that Γ is unique with respect to this property. Let $\Gamma': (A, \beta^1)^\wedge \rightarrow R(A)$ be an algebra isomorphism such that $\Gamma' a = R_a$ for each a in A . For F in $(A, \beta^1)^\wedge$ and a in A the element $\gamma(F, a)$ in A , when considered as an element of $(A, \beta^1)^\wedge$ is equal to $F:a$. Therefore, $(\Gamma' F)(\Gamma' a) = \Gamma'(F:a) = \Gamma'(\gamma(F, a)) = R_{\gamma(F, a)}$ by the algebraic properties of Γ' . Since $\Gamma' F$ is in element of $R(A)$ and so norm-continuous, we have for each a in A that $(\Gamma' F)a = \lim (\Gamma' F)(e_\alpha a) = \lim (\Gamma' F)(R_a e_\alpha) = \lim (\Gamma' F)(\Gamma' a) e_\alpha = \lim R_{\gamma(F, a)} e_\alpha = \lim e_\alpha \gamma(F, a) = \gamma(F, a) = (\Gamma F)a$. Hence, $\Gamma' F = \Gamma F$ for each F in $(A, \beta^1)^\wedge$. The proof is now concluded.

Next, we give necessary and sufficient conditions for $(A, \beta^1)^\wedge$ and $(A, \beta^2)^\wedge$ to be realized as subsets of A^{**} .

4.2.8 Definition. Let $\mathcal{L}(A)$ (resp., $\mathcal{R}(A)$) denote the set of elements F in A^{**} such that Fa (resp., aF) is an element of A (considered as a subset of A^{**}) for each a in A . Let $\mathcal{J}(A)$ denote the intersection of $\mathcal{L}(A)$ and $\mathcal{R}(A)$.

The set $\mathcal{L}(A)$ (resp., $\mathcal{R}(A)$, $\mathcal{J}(A)$) is the maximal subalgebra of A^{**} under the Arens multiplication $(F, G) \rightarrow FG$ in which A is a left (resp., right, two-sided) ideal.

4.2.9 Lemma. The sets $\mathcal{L}(A)$, $\mathcal{R}(A)$, and $\mathcal{J}(A)$ are also subalgebras of A^{**} under the Arens multiplication $(F, G) \rightarrow F:G$.

Proof. Let F and G be elements of $\mathcal{L}(A)$; then $F:G$ is an element of A^{**} . For each a in A we have $(F:G)a = (a:F):G = (Fa):G = G(Fa)$ since Fa is in A . Thus, $F:G$ is an element of $\mathcal{L}(A)$ since $G(Fa)$ is also in A . The set $\mathcal{R}(A)$ is an algebra under this multiplication by a similar argument, and $\mathcal{J}(A) = \mathcal{L}(A) \cap \mathcal{R}(A)$ is also.

4.2.10 Theorem. If A is a Banach algebra with bounded approximate identity, then the following are equivalent:

(i) $A^* = (A, \beta^1)^*$ (resp., $(A, \beta^2)^*$); (ii) if $\mathcal{R}(A)$ (resp., $\mathcal{L}(A)$) is considered as an algebra under the multiplication $(F, G) \rightarrow F:G$ (resp., FG), then as algebras $\mathcal{R}(A) = (A, \beta^1)^\wedge$ (resp., $\mathcal{L}(A) = (A, \beta^2)^\wedge$); and (iii) if $\mathcal{R}(A)$ (resp., $\mathcal{L}(A)$) is considered as an algebra under the multiplication $(F, G) \rightarrow F:G$ (resp., FG), then there exists a unique isometric algebra isomorphism $\Gamma: \mathcal{R}(A) \rightarrow \mathcal{R}(A)$ (resp., $\Gamma: \mathcal{L}(A) \rightarrow \mathcal{L}(A)$) such that $(\Gamma a)x = xa = R_a x$ (resp., $(\Gamma a)x = ax = L_a x$) for a and x in A .

Moreover, if any of these conditions hold, $\mathcal{R}(A)$ (resp., $\mathcal{L}(A)$) is a Banach algebra with identity on which the Arens multiplications are regular.

Proof. To show that (i) \Rightarrow (ii), let F be an element of $\mathcal{R}(A)$; then F is a linear functional on $A^* = (A, \beta^1)^*$.

To show that F is an element of $(A, \beta^1)^\wedge$, it suffices to show that F is $\sigma(A^*, A)$ -continuous at 0 on the polar of each set of the form $U = \{x \in A : \|ax\| \leq 1\}$ for a in A by Theorem 4.1.1, the remarks following Definition 2.1.1, and Lemma 1 of [18, p. 102].

Let $\{f_\alpha\}$ be a net in U^0 converging to 0 in the $\sigma(A^*, A)$ topology. Then by Lemma 4.2.4 there exists a net $\{g_\alpha\}$ in the unit ball of A^* such that $f = g_\alpha \cdot a$ for each α . By the $\sigma(A^*, A)$ -compactness of the unit ball of A^* we can choose g in A^* of norm at most 1 such that it is a $\sigma(A^*, A)$ -cluster point of the net $\{g_\alpha\}$. Then $g \cdot a$ is a $\sigma(A^*, A)$ -cluster point of the net $\{g_\alpha \cdot a\}$; this net is the same as $\{f_\alpha\}$ which converges to 0 in the $\sigma(A^*, A)$ topology; therefore, $g \cdot a = 0$. Since aF is an element of A , we have that $aF(g) = F(g \cdot a) = F(0) = 0$ is a $\sigma(A^*, A)$ -cluster point of the net $\{aF(g_\alpha)\}$, which is the same as the net $\{F(f_\alpha)\}$. Therefore, F is an element of $(A, \beta^1)^\wedge$; and we have shown that $\mathcal{R}(A)$ is a subset of $(A, \beta^1)^\wedge$.

If F is an element of $(A, \beta^1)^\wedge$, then F is a linear functional on $A^* = (A, \beta^1)^*$; and we need to show that F is norm-continuous on A^* . Let $\{f_n\}$ be a sequence in A^* converging to 0 in norm. By hypothesis and Theorem 2.1.7 we see that $A^* = A^* \cdot A$; hence, there exists an element a in A and a sequence $\{g_n\}$ in A^* converging to 0 in norm such that $f_n = g_n \cdot a$ by [14]. We may assume that $\|g_n\| \leq 1$ for

all n . Then each f_n is an element of the polar of the β^1 -neighborhood $\{x \in A: \|ax\| \leq 1\}$ by Lemma 4.2.4. Since F is $\sigma(A^*, A)$ -continuous on such polars, $\lim F(f_n) = \lim F(g_n \cdot a) = 0$; thus F is an element of A^{**} .

Since aF is defined as an element of A^{**} for each a in A and since there exists a unique element b in A such that $f(b) = F(f \cdot a) = aF(f)$ for every f in A^* by Lemma 4.2.5, we see that F is in $\mathcal{R}(A)$. We have proved that $(A, \beta^1)^\wedge = \mathcal{R}(A)$ as sets; clearly, they are equal as algebras.

(ii) \Rightarrow (iii). From the hypothesis and Theorem 4.2.7 there exists a unique algebra isomorphism $\Gamma: \mathcal{R}(A) \rightarrow R(A)$ such that $\Gamma a = R_a$. It suffices to show that Γ is an isometry.

Let F be an element of $\mathcal{R}(A)$ of norm at most 1, and let $\{e_\alpha\}$ be a bounded approximate identity for A . Then F is a $\sigma(A^{**}, A^*)$ -cluster point of the net $\{(\Gamma F)e_\alpha\}$. To see this, we note in Theorem 4.2.7 that if we consider each element $(\Gamma F)e_\alpha$ in A as an element of $\mathcal{R}(A) = (A, \beta^1)^\wedge$, then $(\Gamma F)e_\alpha = F:e_\alpha$. Choose G to be a $\sigma(A^{**}, A^*)$ -cluster point of the net $\{F:e_\alpha\}$ in the unit ball of A^{**} , a $\sigma(A^{**}, A^*)$ -compact set. Then $G:a$ is a $\sigma(A^{**}, A^*)$ -cluster point of the net $\{(F:e_\alpha):a\}$ for each a in A . However, $\lim \|F:(e_\alpha:a) - F:a\| = \lim \|F:(ae_\alpha - a)\| \leq \lim \|ae_\alpha - a\| = 0$; therefore, $G:a = F:a$ for each a in A . Since $(\Gamma G)a = G:a$ and $(\Gamma F)a = F:a$ in $\mathcal{R}(A) = (A, \beta^1)^\wedge$ as remarked above, $\Gamma G = \Gamma F$. Since Γ is

injective, $F = G$ is a $\sigma(A^{**}, A^*)$ -cluster point of the net $\{F: e_\alpha\}$. Now we see that $\|F\| \leq \sup\{\|(\Gamma F)e_\alpha\|\} \leq \sup\{\|(\Gamma F)a\| : a \in A, \|a\| \leq 1\} = \|\Gamma F\| \leq \sup\{\|F:a\| : a \in A, \|a\| \leq 1\} \leq \|F\|$. It follows that Γ is an isometry.

(iii) \Rightarrow (i). Let 0 be an element of A^{**} such that $aG(f) = G(f \cdot a) = 0$ for every f in A^* and a in A . Then G is an element of $\mathcal{R}(A)$ since $aG = G:a = 0$ for each a in A . Moreover, if $\{e_\alpha\}$ is a bounded approximate identity for A and a is any element of A , then $(\Gamma G)a = \lim (\Gamma G)(e_\alpha a) = \lim (\Gamma G)((\Gamma a)e_\alpha) = \lim (\Gamma(G:a))e_\alpha = 0$. Therefore, $\Gamma G = 0$; and since Γ is injective, $G = 0$.

Thus, $A^* \cdot A$ is $\sigma(A^*, A^{**})$ -dense in A^* . Since $(A, \beta^1)^* = A^* \cdot A$ is convex and norm-closed by Theorem 2.1.7, we see that $A^* = (A, \beta^1)^*$.

That $\mathcal{R}(A)$ is a Banach algebra with identity under any of these conditions follows from part (ii) (also, see Theorem 3.1.7). It remains to show that the Arens multiplications are regular on $\mathcal{R}(A)$. If this were not true, there would exist elements F and G in $\mathcal{R}(A)$ and an element f in A^* such that $FG(f) \neq G:F(f)$. By part (i) and Theorem 2.1.7 choose elements a in A and g in A^* such that $f = g \cdot a$. Letting b denote the element aF in A , we would have that $FG(f) = FG(g \cdot a) = (aF)G(g) = G \cdot g(b) = G(g \cdot b) = G(b:g) = G:b(g) = G:(aF)(g) = G:(F:a) = G:F(a:g) = G:F(f \cdot a) = G:F(f)$, a contradiction. The theorem is now concluded.

We now develop results for the β topology.

4.2.11 Lemma. If F is an element of $(A, \beta)^\wedge$, then $F|_{(A, \beta^i)^*}$ is an element of $(A, \beta^i)^\wedge$ for $i=1, 2$.

Proof. Since the polar in A' of each β^i -neighborhood is contained in $(A, \beta)^*$ by Theorem 2.1.7 and [18, p.35], the lemma follows from Theorem 4.1.1.

4.2.12 Theorem. Both Arens multiplications can be defined on $(A, \beta)^\wedge$: that is, FG and $F:G$ are well-defined elements of $(A, \beta)^*$ for F and G in $(A, \beta)^\wedge$.

Proof. We show that the Arens multiplication $(F, G) \rightarrow F:G$ can be defined on $(A, \beta)^\wedge$ by verifying the two conditions of Definition 3.1.1. The proof for the other multiplication is similar.

First, multiplication on A is separately β -continuous by Theorem 2.1.4 so that $(A, \beta)^*$ is a left A -module by Lemma 4.1.4. Next, we must show that $(A, \beta)^*$ is invariant under the mapping $(F, f) \rightarrow F:f$ for F in $(A, \beta)^\wedge$ and f in $(A, \beta)^*$. By Theorem 2.1.7 for each f in $(A, \beta)^*$ there are elements g_i in $(A, \beta^i)^*$ such that $f = g_1 + g_2$. Then $F:f = F:g_1 + F:g_2$, where each term is defined as an element of A' . To show invariance it suffices to show that $F:g_i$ ($i=1, 2$) are elements of $(A, \beta)^*$.

Let F_i denote the restriction of F to $(A, \beta^i)^*$ for $i=1,2$; then F_i is an element of $(A, \beta^i)^\wedge$ by Lemma 4.2.11. The proof of Theorem 4.2.6 shows that for each x in A the elements $x \cdot g_1$ and $F_1: g_1$ are elements of $(A, \beta^1)^*$. Therefore, $F: g_1(x) = F(x \cdot g_1) = F_1(x \cdot g_1) = F_1: g_1(x)$ for each x in A . Hence, $F: g_1$ is an element of $(A, \beta)^*$.

By Theorem 2.1.7 there are elements a in A and h in A^* such that $g_2 = a \cdot h$. Define a linear functional $\gamma(F, h)$ on A by $\gamma(F, h)(x) = F(x \cdot h)$. Moreover, $\gamma(F, h)$ is an element of A^* . To see this, let $\{x_n\}$ be a sequence in A converging to 0 in norm. Then by [] there exist in A an element y and a sequence $\{y_n\}$ converging to 0 in norm such that $x_n = y y_n$. We may assume that $\|y_n\| \leq 1$ for each n . Then $\{y \cdot (y_n \cdot h)\}$ is a subset of the polar of the set $\{x \in A: \|xy\| \leq 1/(\|h\| + 1)\}$ in $(A, \beta^2)^*$ by Lemma 4.2.4. Moreover, the net $\{y \cdot (y_n \cdot h)\}$ converges to 0 in the $\sigma(A^*, A)$ topology. Since F_2 is an element of $(A, \beta^2)^\wedge$, we see that

$$\begin{aligned} \lim \gamma(F, h)(x_n) &= \lim F(x_n \cdot h) = \lim F(y \cdot (y_n \cdot h)) \\ &= \lim F_2(y \cdot (y_n \cdot h)) = 0. \end{aligned}$$

Therefore, $\gamma(F, h)$ is an element of A^* . Consequently, $F: g_2(x) = F(x \cdot g_2) = F(x \cdot (a \cdot h))$
 $= \gamma(F, h)(xa) = a \cdot \gamma(F, h)(x)$; thus, $F: g_2$ is an element of $(A, \beta)^*$ by Theorem 2.1.7. We have now shown that $(A, \beta)^*$ is invariant under the mapping $(F, f) \rightarrow F: f$ for F in $(A, \beta)^\wedge$ and f in $(A, \beta)^*$. Applying Definition 3.1.1, we see that $F: G$ is well-defined as an element of $(A, \beta)^*$ for F and

G in $(A, \beta)^\wedge$.

Assuming that $(A, \beta)^* = A^*$, we show in Theorem 4.2.14 that $(A, \beta)^\wedge$ is an algebra under either Arens multiplication; however, we do not know if this is true without the assumption $A^* = (A, \beta)^*$.

4.2.13 Theorem. If A is a Banach algebra with bounded approximate identity, there exists a vector space isomorphism $\Gamma: (A, \beta)^\wedge \rightarrow M(A)$ such that $\Gamma a = (L_a, R_a)$, where $L_a x = ax$ and $R_a x = xa$ for a and x in A .

Proof. Let F be an element of $(A, \beta)^\wedge$, and let F_i denote the restriction of F to $(A, \beta^i)^*$ for $i = 1, 2$. Then F_i is an element of $(A, \beta^i)^\wedge$ ($i = 1, 2$) by Lemma 4.2.11. Let $F_1 \rightarrow \bar{F}_1$ and $F_2 \rightarrow \bar{F}_2$ denote, respectively, the algebra isomorphisms $(A, \beta^1)^\wedge \rightarrow R(A)$ and $(A, \beta^2)^\wedge \rightarrow L(A)$ defined in Theorem 4.2.7. We wish to show that (\bar{F}_2, \bar{F}_1) is an element of $M(A)$.

It is helpful in this proof to keep in mind that \bar{F}_1 and \bar{F}_2 are defined uniquely by the relations $f(\bar{F}_1(x)) = F_1(f \cdot x)$ and $f(\bar{F}_2(x)) = F_2(x \cdot f)$ for all f in A^* and x in A . Since $a \cdot f \cdot b$ is in $(A, \beta^1)^* \cap (A, \beta^2)^*$ for each f in A^* and a and b in A by Theorem 2.1.7, we have that $f(a \bar{F}_2(b)) = f \cdot a(\bar{F}_2(b)) = F_2(b \cdot f \cdot a) = F_1(b \cdot f \cdot a) = b \cdot f(\bar{F}_1(a)) = f(\bar{F}_1(a)b)$. Thus, $a \bar{F}_2(b) = \bar{F}_1(a)b$ for every a and b

in A since A^* separates points of A . Therefore, (\bar{F}_2, \bar{F}_1) is an element of $M(A)$.

Define $\Gamma: (A, \beta)^\wedge \rightarrow M(A)$ by $\Gamma F = (\bar{F}_2, \bar{F}_1)$. The mapping Γ is easily seen to be well-defined and linear such that $\Gamma a = (L_a, R_a)$ for each a in A . To show that Γ is injective, suppose that $\Gamma F = \Gamma G$. Then $\bar{F}_i = \bar{G}_i$ ($i=1,2$); since $F_i \rightarrow \bar{F}_i$ is an isomorphism, we have that $F_i = G_i$. Moreover, $A^* = (A, \beta)^* = (A, \beta^1)^* + (A, \beta^2)^*$ by hypothesis and Theorem 2.1.7; thus, we see that $F = G$.

Finally, to show that Γ is surjective, let (S, T) be an element of $M(A)$. Then S is a left centralizer and T is a right centralizer by Theorem 4.2.3. Since the set of right centralizers is isomorphic to $(A, \beta^1)^\wedge$ and the set of left centralizers is isomorphic to $(A, \beta^2)^\wedge$ by Theorem 4.2.7, there exists elements F in $(A, \beta^1)^\wedge$ and G in $(A, \beta^2)^\wedge$ such that \bar{F} and \bar{G} (using our notation above) equals T and S respectively. Therefore, if there is a linear functional H in $(A, \beta)^\wedge$ such that $H_1 = F$ and $H_2 = G$, then $\Gamma H = (S, T)$ and Γ is surjective.

In order to define such an H we first show that $F(f) = G(f)$ for every f in $(A, \beta^1)^* \cap (A, \beta^2)^*$. For each such f there exist elements f_1 and f_2 in A^* and elements a and b in A such that $f_1 \cdot a = f = b \cdot f_2$ by Theorem 2.1.7. Let $\{e_\alpha\}$ be a bounded approximate identity for A . Since (\bar{G}, \bar{F}) is in $M(A)$ and \bar{G} is a left

centralizer by Theorem 4.2.3, for each α we have

$$\begin{aligned} f_1(\bar{F}_1(a)e_\alpha) &= f_1(a\bar{G}(e_\alpha)) = f_1 \cdot a(\bar{G}(e_\alpha)) = b \cdot f_2(\bar{G}(e_\alpha)) \\ &= f_2(\bar{G}(e_\alpha)b) = f_2(\bar{G}(e_\alpha b)). \text{ Therefore, } F(f) = F(f_1 \cdot a) \\ &= f_1(\bar{F}(a)) \lim f_1(\bar{F}(a)e_\alpha) = \lim f_2(\bar{G}(e_\alpha b)) = f_2(G(b)) \\ &= f_2(\bar{G}(b)) = G(b \cdot f_2) = G(f) \text{ by our opening remarks and the} \\ &\text{norm-continuity of } \bar{G}. \end{aligned}$$

Now, for each f in $(A, \beta)^*$ there exist elements f_1 in $(A, \beta^1)^*$ such that $f = f_1 + f_2$. Define H on $(A, \beta)^*$ by $H(f) = F(f_1) + G(f_2)$. Then H is well-defined since F and G agree on $(A, \beta^1)^* \cap (A, \beta^2)^*$. Moreover, $H_1 = F$ and $H_2 = G$. It remains to show that H is an element of $(A, \beta)^\wedge$.

A subbase for the β topology consists of all sets of the form $\{x \in A: \|ax\| \leq 1\}$ and of the form $\{x \in A: \|xa\| \leq 1\}$ for a in A by Definition 2.1.1. Since any set of the form $U = \{x \in A: \|ax\| \leq 1\}$ is a β^1 -neighborhood and its polar in $(A, \beta)^*$ is contained in $(A, \beta^1)^*$, we see that H restricted to U° equals F , an element of $(A, \beta^1)^\wedge$; thus H is $\sigma(A^*, A)$ -continuous on U° . Similarly, H is $\sigma(A^*, A)$ -continuous on the polar in $(A, \beta)^*$ of each set of the form $\{x \in A: \|xa\| \leq 1\}$. It is easy to check that this subbase for β satisfies the hypotheses of Corollary 4.1.3; hence, H is an element of $(A, \beta)^\wedge$. We have now shown that Γ is surjective and concluded the proof of the theorem.

4.2.14 Theorem. If A is a Banach algebra with bounded approximate identity, then the following are equivalent:
 (i) $A^* = (A, \beta)^*$; (ii) $(A, \beta)^\wedge$ is an algebra under either Arens multiplication and is equal to $\mathcal{J}(A)$ as an algebra under the same Arens multiplication; and (iii) if $\mathcal{J}(A)$ is considered as an algebra under the multiplication $(F, G) \rightarrow FG$, then there exists a unique isometric algebra isomorphism $\Gamma: \mathcal{J}(A) \rightarrow M(A)$ such that $\Gamma a = (L_a, R_a)$, where $L_a x = ax$ and $R_a x = xa$ for a and x in A .

Moreover, if any of these conditions hold, $\mathcal{J}(A)$ is a Banach algebra with identity on which the Arens multiplications are regular.

Proof. To show that (i) \Rightarrow (ii), let F be an element of $\mathcal{J}(A)$. Then F is a linear functional on $A^* = (A, \beta)^*$. To show that F is an element of $(A, \beta)^\wedge$, it suffices to show that F is $\sigma(A^*, A)$ -continuous at 0 on the polar of each set of the form $U = \{x \in A: \|ax\| \leq 1\}$ and $V = \{x \in A: \|xa\| \leq 1\}$ for a in A by Corollary 4.1.2, Definition 2.1.1, and Lemma 1 of [18, p. 102].

Since U is a β^1 -neighborhood, the argument that F is $\sigma(A^*, A)$ -continuous 0 on U° is the same as that which occurs in the second paragraph of the proof of Theorem 4.2.10. Analogously, F is also $\sigma(A^*, A)$ -continuous in V° . Therefore, $\mathcal{J}(A)$ is contained in $(A, \beta)^\wedge$.

If F is an element of $(A, \beta)^\wedge$, then $F_i = F|_{(A, \beta_i)^*}$ ($i=1,2$) is an element of $(A, \beta^i)^\wedge$ by Lemma 4.2.11. The right (resp., left) centralizer \bar{F}_1 (resp., \bar{F}_2) defined in Theorem 4.2.7 is norm-continuous and (\bar{F}_2, \bar{F}_1) is an element of $M(A)$ by Theorem 4.2.13. Recall that the values of \bar{F}_2 and \bar{F}_1 at a in A are defined uniquely by the relations $f(\bar{F}_2(a)) = F_2(f \cdot a)$ and $f(\bar{F}_1(a)) = F_1(a \cdot f)$ for every f in A^* .

Let $\{e_\alpha\}$ be a bounded approximate identity for A and let S denote the unit ball of A^{**} . Choose G to be a $\sigma(A^{**}, A^*)$ -cluster point of the net $\{\bar{F}_1(e_\alpha)\}$ in the $\sigma(A^{**}, A^*)$ -compact set $\|\bar{F}_1\|S$. Then $G(f \cdot a)$ is a cluster point of the net $\{f \cdot a(\bar{F}_1(e_\alpha))\}$ for each f in A^* and a in A . However, $f \cdot a(\bar{F}_1(e_\alpha)) = f(a\bar{F}_1(e_\alpha)) = f(\bar{F}_1(ae_\alpha))$ for each α , and thus $\lim f(\bar{F}_1(ae_\alpha)) = f(\bar{F}_1(a)) = F_1(f \cdot a) = F(f \cdot a)$. Therefore, $G(f \cdot a) = F(f \cdot a)$ for every f in A^* and a in A .

Moreover, $G(a \cdot f)$ is a cluster point of the net $\{a \cdot f(\bar{F}_1(e_\alpha))\}$ for each f in A^* and a in A ; yet, $a \cdot f(\bar{F}_1(e_\alpha)) = f(\bar{F}_1(e_\alpha)a) = f(e_\alpha \bar{F}_2(a))$ for every α and $\lim f(e_\alpha \bar{F}_2(a)) = f(\bar{F}_2(a)) = F_2(a \cdot f) = F(a \cdot f)$. Thus, we have that G , an element of A^{**} , is equal to F on the set $A \cdot A^* + A^* \cdot A$, which equals A^* by hypothesis and Theorem 2.1.7. Hence, every element of $(A, \beta)^\wedge$ is an element of A^{**} .

For F in $(A, \beta)^\wedge$ and a in A we now see that aF

is defined as an element of A^{**} and that $aF(f) = F(f \cdot a) = F_1(f \cdot a) = f(\bar{F}_1(a))$ for every f in A^* . Hence, aF is actually an element of A . That Fa is also an element of A is shown similarly; thus, $(A, \beta)^\wedge = \mathcal{A}(A)$. Since either Arens multiplication is defined on $(A, \beta)^\wedge$ by Theorem 4.2.12 and it agrees with the corresponding Arens multiplication on $\mathcal{A}(A)$, it is clear that $(A, \beta)^\wedge$ must be an algebra under either multiplication since $\mathcal{A}(A)$ is by Lemma 4.2.9.

(ii) \Rightarrow (iii). From the hypothesis and Theorem 4.2.13 there exists a vector space isomorphism $\Gamma: \mathcal{A}(A) \rightarrow M(A)$ such that $\Gamma a = (L_a, R_a)$ for every a in A . For F in $\mathcal{A}(A)$ the mapping Γ is defined such that $\Gamma F = (S, S')$, where Sa and $S'a$ considered as elements of A^{**} equal Fa and aF respectively for each a in A .

Let G be an element of $\mathcal{A}(A)$ such that $\Gamma G = (T, T')$ and denote $\Gamma(FG)$ by (U, U') . Therefore, in A^{**} we have that $Ua = (FG)a = F(Ga) = S(Ta) = (ST)a$ and $U'a = a(FG) = (aF)G = T'(S'a) = (T'S')a$. Thus, $\Gamma(FG) = (\Gamma F)(\Gamma G)$ and Γ is an algebra homomorphism.

To show that Γ is an isometry, recall that $\|\Gamma F\| = \|(S, S')\| = \|S\| = \|S'\|$. For each a in A the norm of the element $S'a$ in A is the same as its norm in A^{**} . Hence, $\|\Gamma F\| \leq \|F\|$. Now let $\{e_\alpha\}$ be a bounded approximate identity for A . Then each $S'e_\alpha$ in A^{**} equals $e_\alpha F = F:e_\alpha$,

and F is a $\sigma(A^{**}, A^*)$ -cluster point of the net $\{F: e_\alpha\}$ in A^{**} . The argument to show this is the same as in Theorem 4.2.10 where we showed the algebra isomorphism $\mathcal{R}(A) \rightarrow R(A)$ is an isometry. Therefore, $\|F\| \leq \sup\{\|e_\alpha F\|\} = \sup\|S'e_\alpha\| \leq \|S'\| = \|\Gamma F\|$.

It remains to show that Γ is unique with respect to the property that $\Gamma a = (L_a, R_a)$ for every a in A . However, the argument is modeled on the proof in Theorem 4.2.7 that the algebra isomorphism $\mathcal{R}(A) \rightarrow R(A)$ is unique with respect to the property that $a \mapsto R_a$ and its left analogue and so it will not be given.

(iii) \Rightarrow (i). Let G be an element of A^{**} such that $G(f \cdot a + b \cdot g) = 0$ for every a and b in A and every f and g in A^* . Then G is an element of $\mathcal{J}(A)$ since $aG = 0 = Ga$ for every a in A . If $\Gamma G = (S, S')$, then $(SL_a, R_a S') = (\Gamma G)(\Gamma a) = \Gamma(Ga) = 0$; in particular, if $\{e_\alpha\}$ is a bounded approximate identity for A , then $S(e_\alpha a) = SL_{e_\alpha}(a) = 0$ and $Sa = \lim S(e_\alpha a) = 0$. It is now immediate that $\Gamma G = 0$ in $M(A)$. Therefore, G is 0 since Γ is injective. Thus, $A^* \cdot A + A \cdot A^*$ is $\sigma(A^*, A^{**})$ -dense in A^* . Since $(A, \beta)^* = A^* \cdot A + A \cdot A^*$ is convex and norm-closed by Theorem 2.1.7, we see that $A^* = (A, \beta)^*$.

Finally, if any of the conditions of the theorem hold, $\mathcal{J}(A)$ is a Banach algebra with identity by part (ii). Moreover, the Arens multiplications on $\mathcal{J}(A)$ are regular. If this

were not true, there would exist elements F and G in $\mathcal{J}(A)$ and an element f of A^* such that $FG(f) \neq G:F(f)$. However, by part (i) and Theorem 2.1.7 there would exist elements a and b in A and elements g and h in A^* such that $f = g \cdot a + b \cdot h$. Then since F and G are in $\mathcal{J}(A)$, by letting x and y denote the elements $aF = F:a$ and $Gb = b:G$ in A , respectively, we would have that

$$\begin{aligned} FG(f) &= FG(g \cdot a) + FG(b \cdot h) = aF(G \cdot g) + F((Gb) \cdot h) = G(g \cdot x) + F:h(y) \\ &= G(x:g) + b:G(F:h) = G:x(g) + b:(G:F)(h) = G:F(a:g) + (G:F):h(b) \\ &= G:F(g \cdot a) + G:F(b \cdot h) = G:F(h), \text{ a contradiction.} \end{aligned}$$

This concludes the theorem.

4.2.15 Example. If G is a locally compact abelian group, then $A = L^1(G)$ is a Banach $*$ -algebra with bounded approximate identity. However, $(A, \beta)^*$ equals A^* if and only if G is discrete: that is, if and only if $L^1(G)$ has an identity in which case the β topology and the norm topology coincide. To see this, we note the well-known result that $L^1(G) * L^\infty(G) = C_u(G)$, the set of all bounded uniformly continuous functions on G . Moreover, that $(A, \beta)^* = L^1(G) * L^\infty(G)$ is easily checked. It is then clear that if $C_u(G) = L^\infty(G)$, then G is discrete. The converse is obvious.

4.2.16 Lemma. If A is a B^* -algebra, then the β^1, β^2 , and β topological duals of A are the same and equal A^* .

Proof. This result follows from Theorem 2.2.3 and the Mackey-Arens Theorem.

4.2.17 Theorem. If A is a B^* -algebra, the following are equivalent: (i) A is dual (that is, A is isomorphic to a B^* -subalgebra of the compact operators on some Hilbert space); (ii) the mapping $L_a: A \rightarrow A$ defined by $L_a x = ax$ is weakly compact for each a in A (that is, each set $\{L_a x: x \in A, \|x\| \leq 1\}$ is relatively $\sigma(A, A^*)$ -compact; (iii) the mapping $R_a: A \rightarrow A$ defined by $R_a x = xa$ is weakly compact for each a in A ; (iv) if S_0 denotes the unit ball of A^* , then $a \cdot S_0$ is relatively $\sigma(A^*, A^{**})$ -compact for each a in A ; (v) the set $S_0 \cdot a$ is relative $\sigma(A^*, A^{**})$ -compact for each a in A ; (vi) $\mathcal{J}(A) = A^{**}$ (that is, A is a two-sided ideal in A^{**}); and (vii) there exists an isometric algebra isomorphism $\Gamma: A^{**} \rightarrow M(A)$ such that $\Gamma_a = (L_a, R_a)$ for a in A .

Proof. For the equivalence of (i), (ii), and (iii) the reader is referred to [7, p.99]. To show (iii) \Rightarrow (iv), note that the mapping $R_a^*: A^* \rightarrow A^*$ defined by $R_a^*(f) = a \cdot f$ is adjoint to R_a , and then apply the Generalized Gantmacher Theorem (see Theorem 1.1.10).

Assume that (iv) holds. The mapping $f \mapsto f^*$ on A^* is $\sigma(A^*, A^{**})$ -continuous; moreover, the unit ball of A^* is invariant under this mapping, and $(f \cdot a)^*(x) = \overline{f \cdot a(x^*)} = \overline{f(ax^*)}$

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$= f^*(xa^*) = a^* \cdot f^*(x)$ for every f in A^* and a and x in A . Therefore, each $S_0 \cdot a$ is relatively $\sigma(A^*, A^{**})$ -compact from the hypothesis.

Assume now that (v) holds, and we shall show that $\mathcal{J}(A) = A^{**}$. For each a in A the mapping $L_a^*: A^* \rightarrow A^*$ defined by $L_a^*(f) = f \cdot a$ is adjoint to L_a , and the mapping $L_a^{**}: A^{**} \rightarrow A^{**}$ defined by $L_a^{**}(F) = aF$ is adjoint to L_a^* . Since L_a^* is weakly compact by hypothesis, the range of L_a^{**} is contained in A (considered as a subset of A^{**}) by the Generalized Gantmacher Theorem. That is, aF and a^*F^* are elements of A for each F in A^{**} and a in A . Moreover, $(a^*F^*)^* = Fa$ is also in A since A^{**} is a B^* -algebra by Theorem 3.2.16. Therefore, A^{**} coincides with $\mathcal{J}(A)$.

That (vi) \Rightarrow (vii) follows from Lemma 4.2.16 and Theorem 4.2.14. Finally, assume (vii); we must prove that L_a is weakly compact for each a in A . By Lemma 4.2.16 and Theorem 4.2.14 there exists an isometric algebra isomorphism $\Omega: M(A) \rightarrow \mathcal{J}(A)$ such that $\Omega(L_x, R_x) = x$ for every x in A . For each F in A^{**} and a in A we have that $\Omega\Gamma(aF) = \Omega((\Gamma a)(\Gamma F)) = \Omega((L_a, R_a)(\Gamma F)) = a\Omega(\Gamma F)$ which is in A since the range of Ω is $\mathcal{J}(A)$. Therefore, each aF must be an element of A since $\Omega\Gamma(x) = \Omega(L_x, R_x) = x$ for each x in A and $\Omega \circ \Gamma$ is an isomorphism. Hence $L_a^{**}: A^{**} \rightarrow A^{**}$, the mapping adjoint to the adjoint mapping of L_a , has its range contained in A . Thus, L_a is weakly compact by the

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Generalized Gantmacher Theorem. This concludes the proof.

Section 3. The μ -Completions of a Banach Algebra with
Bounded Approximate Identity.

The completions with respect to μ^1 and μ^2 of a Banach algebra A with bounded approximate identity are algebraic under certain Arens multiplications and can be characterized as particular sets of linear functionals in $A^{*'}.$ If the Arens multiplications are regular on $A^{**},$ then A^{**} is a subset of both $(A, \mu^1)^\wedge$ and $(A, \mu^2)^\wedge$; moreover, the μ^1 and μ^2 completion topologies restricted to A^{**} coincide respectively with the μ_e^1 and μ_e^2 topologies on $A^{**}.$

The μ -completion of A is the intersection of $(A, \mu^1)^\wedge$ and $(A, \mu^2)^\wedge,$ and both Arens multiplications can be defined on $(A, \mu)^\wedge.$ Under the assumption that A is a commutative B^* -algebra we prove that $(A, \mu)^\wedge$ equals A^{**} and, thus, is an algebra under either Arens multiplication.

In this section A denotes a Banach algebra with bounded approximate identity; thus, the $\mu^1, \mu^2,$ and μ topologies on A are Hausdorff. Recall that the topological duals of A with respect to each of these topologies are the same and equal $A^*.$ We now prove that $(A, \mu^1)^\wedge$ and $(A, \mu^2)^\wedge$ are algebras under certain Arens multiplications and can be characterized as particular subsets of $A^{*'}.$

4.3.1 Lemma. If f is an element of A^* and $U = \{x \in A : \|f \cdot x\| \leq 1\}$, then $U^0 = \{F \cdot f : F \in A^{**}, \|F\| \leq 1\}$. If $V = \{x \in A : \|x \cdot f\| \leq 1\}$, then $V^0 = \{F : f : F \in A^{**}, \|F\| \leq 1\}$. Moreover, $|f(x)| \leq \min \{\|f \cdot x\|, \|x \cdot f\|\}$ for every x in A ; therefore, f is in $U^0 \cap V^0$.

Proof. Let g be an element of U^0 . Define a linear functional H on the subspace $f \cdot A = \{f \cdot x : x \in A\}$ of A^* by $H(f \cdot x) = g(x)$. Since $|g(x)| \leq \|f \cdot x\|$ for all x in A [18, p. 13], the linear functional H is well-defined and bounded by 1 on $f \cdot A$; therefore, by the Hahn-Banach Theorem it can be extended to a bounded linear functional \bar{H} on A^* such that $\|\bar{H}\| \leq 1$ and $\bar{H} \cdot f = g$. Thus, U^0 is contained in the set $\{F \cdot f : F \in A^{**}, \|F\| \leq 1\}$, and the reverse containment is clear. The proof for V^0 is similar.

Finally, if $\{e_\alpha\}$ is a bounded approximate identity for A , then $\lim f(xe_\alpha) = \lim f(e_\alpha x)$ for every x in A . Then for $\epsilon > 0$ and x in A there is an element e of the bounded approximate identity such that $|f(x)| < \epsilon + \min \{f(xe), f(ex)\} \leq \epsilon + \min \{\|f \cdot x\|, \|x \cdot f\|\}$. Since ϵ is arbitrary, the desired inequality is true.

4.3.2 Lemma. For F in $(A, \mu^1)^\wedge$ (resp., $(A, \mu^2)^\wedge$) and f in A^* the linear functional $F : f$ (resp., $F \cdot f$) on A is an element of A^* . Moreover, $G(F : f) = F(G \cdot f)$ (resp., $G(F \cdot f) = F(G : f)$) for every G in A^{**} .

Proof. If $\{x_\alpha\}$ is a net in the unit ball of A converging to 0 in the $\sigma(A, A^*)$ topology, then the set $\{x_\alpha \cdot f\}$ is contained in the polar of the μ^1 -neighborhood $\{x \in A: \|f \cdot x\| \leq 1\}$ by Lemma 4.3.1. Moreover, the net $\{x_\alpha \cdot f\}$ converges to 0 in the $\sigma(A^*, A)$ topology. Therefore, $\lim F: f(x_\alpha) = \lim F(x_\alpha \cdot f) = 0$ since F is an element of $(A, \mu^1)^\wedge$. Since A is a Banach space, the linear functional $F: f$ is norm-continuous on A [18, Theorem 5, p.111; Theorem 1, p.26].

For any non-zero G in A^{**} choose a net $\{a_\alpha\}$ in A converging to G in the $\sigma(A^{**}, A^*)$ topology such that $\|a_\alpha\| \leq \|G\|$ by Goldstine's Theorem. Then the set $\{G \cdot f\} \cup \{a_\alpha \cdot f\}$ is contained in the polar of the μ^1 -neighborhood $\{x \in A: \|f \cdot x\| \leq 1/\|G\|\}$ by Lemma 4.3.1. Moreover, $G \cdot f$ is the $\sigma(A^*, A)$ -limit of the net $\{a_\alpha \cdot f\}$. Hence, $F(G \cdot f) = \lim F(a_\alpha \cdot f) = \lim F: f(a_\alpha) = G(F: f)$ since F is an element of $(A, \mu^1)^\wedge$.

The proof for F in $(A, \mu^2)^\wedge$ is analogous.

4.3.3 Theorem. The completion of A with respect to μ^1 (resp., μ^2) is an algebra under the Arens multiplication $(F, G) \rightarrow F: G$ (resp., FG).

Proof. We show that $(A, \mu^1)^\wedge$ is an algebra by verifying the hypotheses of Theorem 4.1.5. First, multiplication on A is separately μ^1 -continuous by Theorem 2.1.4. Next, $(A, \mu^1)^* = A^*$ is invariant under the mapping

$(F, f) \rightarrow F:f$ for F in $(A, \mu^1)^\wedge$ and f in A^* by Lemma 4.3.2. The remaining hypothesis of Theorem 4.1.5 to be satisfied is that for each F in $(A, \mu^1)^\wedge$ and each μ^1 -neighborhood U there is a μ^1 -neighborhood V such that $F:U^\circ$ is contained in V° .

The collection of sets of the form $\{x \in A: \|f \cdot x\| \leq 1\}$ for f in A^* is a subbase for the μ^1 topology which satisfies the hypotheses of Corollary 4.1.3. From the proof of that corollary we see that the polar of any μ^1 -neighborhood U is contained in a set $W_1^\circ + \dots + W_n^\circ$, where each W_i is a μ^1 -neighborhood of the form $\{x \in A: \|f_i \cdot x\| \leq 1\}$ for f_i in A^* .

Now, for F in $(A, \mu^1)^\wedge$ the set $F:U^\circ$ is contained in $F:W_1^\circ + \dots + F:W_n^\circ$. Moreover, since each linear functional $F:f_i$ is an element of A^* , each set $F:W_i^\circ$ is contained in the polar of the μ^1 -neighborhood $V_i = \{x \in A: \|(F:f_i) \cdot x\| \leq 1\}$. To see this, let g be an element of W_i° ; then $g = H \cdot f_i$ for some H in A^{**} such that $\|H\| \leq 1$ by Lemma 4.3.1. Moreover, for x in A we have that $F:g(x) = F:(H \cdot f_i)(x) = F(x \cdot (H \cdot f_i)) = F((x \cdot H) \cdot f_i) = x \cdot H(F:f_i) = H((F:f_i) \cdot x)$ by Lemma 4.3.2. Therefore, $F:g = H(F:f_i)$ is an element of V_i° by Lemma 4.3.1.

Let V equal $\cap \{(1/n)V_i: i = 1, \dots, n\}$; then V is a μ^1 -neighborhood, and it is easily verified that $V_1^\circ + \dots + V_n^\circ$ is a subset of V° . Hence, the subset $F:U^\circ$ of

$V_1^0 + \dots + V_n^0$ is contained in V^0 .

Applying Theorem 4.1.5, we have that $(A, \mu^1)^\wedge$ is an algebra under the Arens multiplication $(F, G) \mapsto F:G$ and that the mapping $F \mapsto F:G$ on $(A, \mu^1)^\wedge$ is continuous in the completion topology for each G in $(A, \mu^1)^\wedge$. The proof for $(A, \mu^2)^\wedge$ is similar.

4.3.4 Definition. For F in $A^{*'}$ and G in A^{**} , define linear functionals FG and $F:G$ on A^* by $FG(f) = F(G \cdot f)$ and $F:G(f) = F(G:f)$. The mappings $(F, G) \mapsto FG$ and $(F, G) \mapsto F:G$ are bilinear operators from $A^{*'} \times A^{**}$ to $A^{*'}$ which extends the Arens multiplications on A^{**} .

4.3.5 Theorem. An element F in $A^{*'}$ is in $(A, \mu^1)^\wedge$ (resp., $(A, \mu^2)^\wedge$) if and only if the mapping $G \mapsto FG$ (resp., $G \mapsto F:G$) from A^{**} to $A^{*'}$ is $\sigma(A^{**}, A^*) - \sigma(A^{*'}, A^*)$ continuous.

Proof. We prove the theorem for $(A, \mu^1)^\wedge$ since the arguments are similar. If F is an element of $(A, \mu^1)^\wedge$, then $F:f$ is an element of A^* for every f in A^* and $G(F:f) = FG(f)$ for every G in A^{**} by Lemma 4.3.2 and Definition 4.3.4. It is now clear that the mapping $G \mapsto FG$ from A^{**} to $A^{*'}$ is $\sigma(A^{**}, A^*) - \sigma(A^{*'}, A^*)$ continuous.

Conversely, let F be an element of $A^{*'}$ such that the mapping $G \mapsto FG$ from A^{**} to $A^{*'}$ is $\sigma(A^{**}, A^*) - \sigma(A^{*'}, A^*)$ continuous. Then it suffices by Corollary 4.1.3 to prove

that F is $\sigma(A^*, A)$ -continuous on the polar of each μ^1 -neighborhood of the form $\{x \in A: \|f \cdot x\| \leq 1\}$ for f in A^* .

Let g be an element of A^* , let U be the set $\{x \in A: \|g \cdot x\| \leq 1\}$, and let $\{f_\alpha\}$ be a net in U^0 converging to 0 in the $\sigma(A^*, A)$ topology. Then there exists a net $\{H_\alpha\}$ in the unit ball of A^{**} such that $f_\alpha = H_\alpha \cdot g$ for every α by Lemma 4.3.1. Since the unit ball of A^{**} is $\sigma(A^{**}, A^*)$ -compact, choose H to be a $\sigma(A^{**}, A^*)$ -cluster point of the net $\{H_\alpha\}$. Then $H \cdot g$ is a $\sigma(A^*, A)$ -cluster point of the net $\{H_\alpha \cdot g\}$, which converges to 0 in the $\sigma(A^*, A)$ topology. Therefore, $H \cdot g$ equals 0 . Moreover, by hypothesis $FG(g) = F(H \cdot g) = 0$ is a cluster point of the net $\{FH_\alpha(g)\}$, where $F(f_\alpha) = F(H_\alpha \cdot g) = FH_\alpha(g)$ for each α . Thus, F is an element of $(A, \mu^1)^\wedge$.

We now give partial results for $(A, \mu)^\wedge$.

4.2.6 Lemma. The completion of A with respect to μ equals $(A, \mu^1)^\wedge \cap (A, \mu^2)^\wedge$ in $A^{*'}$.

Proof. This follows directly from Corollary 4.1.3 and the fact that the collection of μ^1 -neighborhoods of the form $\{x \in A: \|f \cdot x\| \leq 1\}$ and of μ^2 -neighborhoods of the form $\{x \in A: \|x \cdot f\| \leq 1\}$ for f in A^* is a subbase for μ satisfying the hypotheses of Corollary 4.1.3.

We have the immediate corollary that the Arens

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multiplications $(F,G) \rightarrow FG$ and $(F,G) \rightarrow F:G$ can be defined on $(A,\mu)^\wedge$ (see Definition 3.1.1). However, we do not know if it is an algebra under either (or, equivalently, both) Arens multiplications; and if it is, whether or not the Arens multiplications are regular. Also, we do not know about the general relationships between $(A,\mu)^\wedge$ and A^{**} as subsets of A^* .

4.3.7 Theorem. If A is a Banach algebra with bounded approximate identity, then (i) A^{**} is a subset of $(A,\mu)^\wedge$ if and only if the Arens multiplications on A^{**} are regular; and (ii) if $(A,\mu)^\wedge$ is a subset of A^{**} , then $(A,\mu)^\wedge$ is an algebra under both Arens multiplications and these multiplications are regular.

Proof. Assume that A^{**} is a subset of $(A,\mu)^\wedge$. If $\{F_\alpha\}$ is any net in the unit ball S of A^{**} , a $\sigma(A^{**}, A^*)$ -compact set, choose H to be some $\sigma(A^{**}, A^*)$ -cluster point of this net in S . For each G in A^{**} and f in A^* we have that $G:F(f)$ and $GF(f)$ are cluster points of the nets $\{G:F_\alpha(f)\}$ and $\{GF_\alpha(f)\}$, respectively, by Lemma 4.3.6 and Theorem 4.3.5. Therefore, the sets $S:f$ and $S \cdot f$ are $\sigma(A^*, A^{**})$ -compact for each f in A^* . That the Arens multiplications on A^{**} are regular follows from Theorem 3.1.8.

Conversely, suppose that the Arens multiplications are regular on A^{**} . If $\{G_\alpha\}$ is a net in A^{**} converging to 0

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in the $\sigma(A^{**}, A^*)$ topology, then for each F in A^{**} and f in A^* we have that $\lim F G_\alpha(f) = \lim G_\alpha : F(f) = 0$ and $\lim f : G_\alpha(f) = \lim G_\alpha F(f) = 0$ by Lemma 3.1.3. Therefore, each F in A^{**} is an element of $(A, \mu)^\wedge$ by Lemma 4.3.6 and Theorem 4.3.5.

To prove (ii), let F and G be elements of $(A, \mu)^\wedge$; then $F:G$ and GF are defined as elements of A^{**} . Choose a net $\{a_\alpha\}$ in A converging to G in the $\sigma(A^{**}, A^*)$ topology by Goldstine's Theorem. Then since F is in $(A, \mu^1)^\wedge$ by Lemma 4.3.6, for each f in A^* we have that $F:G(f) = \lim F:a_\alpha(f) = \lim a_\alpha F(f) = GF(f)$ by Theorem 4.3.5 and Lemma 3.1.3. Therefore, since $F:G$ is an element of $(A, \mu^1)^\wedge$ and GF is an element of $(A, \mu^2)^\wedge$ by Lemma 4.3.6 and Theorem 4.3.3, the element $F:G = GF$ is in $(A, \mu)^\wedge$ by Lemma 4.3.6; and part (ii) now follows.

4.3.8 Corollary. If A^{**} is a subset of $(A, \mu)^\wedge$, then the restrictions of the μ^1, μ^2 , and μ completion topologies to A^{**} coincide with the μ_e^1, μ_e^2 , and μ_e topologies, respectively.

Proof. Let $\{F_\alpha\}$ be a net in A^{**} converging to 0 in the μ_e^1 topology. Then $\lim \|F_\alpha : f\| = 0$ for each f in A^* by Definition 3.1.6. Any element f in the polar of the μ^1 -neighborhood $V = \{x \in A : \|g \cdot x\| \leq 1\}$ can be written as $f = H \cdot g$, where H is an element of A^{**} of norm at most 1

by Lemma 4.3.1. Therefore, $\lim |F_\alpha H(g)| = \lim |H:F_\alpha(g)| \leq \lim \|F_\alpha:g\| = 0$ since the Arens multiplications are regular by Theorem 4.3.7. Hence, there exists α_0 such that the set $\{f_\alpha:\alpha>\alpha_0\}$ is contained in $V^{\circ\circ} \cap (A,\mu)^\wedge$, and so the net $\{F_\alpha\}$ converges to 0 in the μ^1 -completion topology.

Conversely, if $\{G_\alpha\}$ is a net in A^{**} converging to 0 in the μ^1 -completion topology and f is any element of A^* , for $\epsilon > 0$ there is α_0 such that the set $\{g_\alpha:\alpha>\alpha_0\}$ is contained in the bipolar of the μ^1 -neighborhood $U = \{x \in A: \|f \cdot x\| \leq \epsilon\}$. In particular, if x is any element of the unit ball of A , then $|G_\alpha:f(x)| = |G_\alpha(x \cdot f)| \leq \epsilon$ whenever $\alpha > \alpha_0$ by Lemma 4.3.1. Consequently, $\lim \|G_\alpha:f\| = 0$ for each f in A^* ; thus, the net $\{G_\alpha\}$ converges to 0 in the μ_e^1 topology.

The proof for the μ_e^2 topology is similar; and note that as a consequence of Corollary 4.1.3 a subbase for the μ -completion topology consists of bipolars (in A^{**}) of μ^1 -neighborhoods and μ^2 -neighborhoods of A restricted to $(A,\mu)^\wedge$ so that the conclusion holds for the μ_e topology as well.

A generalization of the next theorem is given in Theorem 5.2.10.

4.3.9 Theorem. If A is a commutative B^* -algebra, then $A^{**} = (A,\mu)^\wedge$.

Proof. We may regard A as $C_0(S)$, the bounded continuous complex-valued functions on a locally compact Hausdorff space which vanish at infinity (see Example 3.3.7 and [7, p.9]). Then A^* is the set of complex regular Borel measures on S [19, p.131]. Since the Arens multiplications are regular on A^{**} by Theorem 3.2.5, we have that A^{**} is a subset of $(A, \mu)^\wedge$ by Theorem 4.3.7.

If F is an element of $(A, \mu)^\wedge$, we wish to show that F is a norm-continuous linear functional on A^* . It suffices to show that if $\{f_n\}$ is a sequence of A^* such that $\lim \|f_n\| = 0$, then $\{F(f_n)\}$ clusters at 0. By choosing a subsequence if necessary, we may assume that $\|f_n\| \leq 1/2^n$ for each n . If $|f_n|$ denotes the total variation measure of f_n , then $\theta = \sum_{n=1}^{\infty} |f_n|$ is a positive measure in A^* . Note that $|f_n(x)| \leq |f_n|(|x|) \leq \theta(|x|) \leq \|\theta \cdot |x|\| = \|\theta \cdot x\|$ for each x in A by Lemma 4.3.1 and a consequence of the Radon-Nikodym Theorem [19, p.126]. Therefore, the sequence $\{f_n\}$ is contained in the polar of the μ -neighborhood $\{x \in A : \|\theta \cdot x\| \leq 1\}$ by Lemma 4.3.1. Since F is an element of $(A, \mu)^\wedge$ and $\lim \|f_n\| = 0$, we see that $\lim F(f_n) = 0$; therefore, F is an element of A^{**} . We have now shown that $A^{**} = (A, \mu)^\wedge$.

Section 4. The \mathcal{A} -Completions of Certain Banach
*-Algebras.

In this section A will denote a Banach $*$ -algebra with bounded approximate identity such that the positive linear functionals on A separate points of A . We then show that $(A, \mathcal{A}^1)^\wedge$ and $(A, \mathcal{A}^2)^\wedge$ are equal to $(A, \mathcal{A})^\wedge$. Moreover, the \mathcal{A} -completion of A is in a natural way a $*$ -algebra under either Arens multiplication; and with respect to the \mathcal{A} -completion topology $(A, \mathcal{A})^\wedge$ is algebraically and topologically isomorphic to the bidual of the $B*$ -algebra enveloping A with the \mathcal{A}_e topology.

We denote the set of positive linear functionals on A by P : that is, $P = \{\theta \in A: \theta(a^*a) \geq 0, \forall a \in A\}$. Since each element of P is norm-continuous by [28], we have that the topologies \mathcal{A}^1 , \mathcal{A}^2 , and \mathcal{A} are Hausdorff on A by the remarks following Definition 2.1.3. Moreover, the topological duals with respect to each of these topologies are the same and equal the span of P by Theorem 2.1.8. We shall now prove that $(A, \mathcal{A}^1)^\wedge = (A, \mathcal{A})^\wedge$.

4.4.1 Definition. For a in A define $|a|$
 $= \sup \{ \theta(a^*a)^{\frac{1}{2}} : \theta \in P, \|\theta\| \leq 1 \}$. Then the mapping $a \mapsto |a|$ is a norm on A since we are assuming that P separates points of A [7, p.23]. Let $C(A)$ denote the completion of A with respect to the norm $a \mapsto |a|$ on A . Then $C(A)$ is a

B^* -algebra and the natural injection $\tau: A \rightarrow C(A)$ is norm-decreasing [7, p.41]. The algebra $C(A)$ is called the B^* -algebra enveloping A . If P' denotes the set $\{\varphi \in C(A) : \varphi(c^*c) \geq 0, \forall c \in C(A)\}$, then each element of P' is norm-continuous on $C(A)$ since $C(A)$ has a bounded approximate identity [28; 7, p. 15]. Moreover, $C(A)^*$ is the span of P' [7, p. 40]. We shall denote the norm of f in $C(A)^*$ by $|f|$. (Compare with Theorem 2.1.8 and its corollary).

4.4.2 Lemma. To each θ in P there corresponds a unique element $\bar{\theta}$ in P' such that $\theta = \bar{\theta} \circ \tau$ and $|\bar{\theta}| = \|\theta\|$. Moreover, $\theta \mapsto \bar{\theta}$ is a bijection from P to P' . Therefore, there is a well-defined linear bijection $f \mapsto \bar{f}$ from $\langle P \rangle$ to $C(A)^*$ such that if $f = \sum_{i=1}^4 \lambda_i \theta_i$, for θ_i in P and λ_i complex numbers, then $\bar{f} = \sum_{i=1}^4 \lambda_i \bar{\theta}_i$.

Proof. The reader is referred to [7, p.42].

4.4.3 Lemma. If θ is an element of P , let $U = \{a \in A : \theta(a^*a) \leq 1\}$ and let $V = \{c \in C(A) : \bar{\theta}(c^*c) \leq 1\}$. Then $V^0 = \{\bar{f} \in C(A)^* : f \in U^0\}$. Furthermore, the mapping $f \mapsto \bar{f}$ from U^0 to V^0 is $\sigma(\langle P \rangle, A) - \sigma(C(A)^*, C(A))$ bicontinuous.

Proof. Let f be a non-zero element of U^0 , let v be an element of V , and let $\epsilon > 0$. Set δ equal to $\min \{1, \epsilon/2 \cdot (\|\theta\| + |\bar{f}|^2) \cdot (2|v| + 1)\}$. Then there is an element

a in A such that $|v - \tau a| < \delta$. Thus, $|\bar{f}(v)^2 - f(\tau a)^2|$
 $= |\bar{f}(v) + \bar{f}(\tau a)| \cdot |f(v) - f(\tau a)| \leq |\bar{f}|^2 |v + \tau a| |v - \tau a|$
 $\leq \delta |\bar{f}|^2 (2|v| + \delta) < \epsilon/2$ and $|\bar{\theta}(\tau a * \tau a) - \bar{\theta}(v * v)|$
 $\leq |\bar{\theta}| (|\tau a * a - (\tau a *) v| + |(\tau a *) v - v * v|) \leq \|\theta\| \tau a - v| (|\tau a| + |v|)$
 $\leq \delta \|\theta\| (2|v| + \delta) < \epsilon/2$. Since f is an element of U^0 , we
have that $|f(x)|^2 \leq \theta(x * x)$ for all x in $[18, p.13]$; and
we see that $|\bar{f}(v)|^2 \leq |\bar{f}(\tau a)|^2 + \epsilon/2 = |f(a)|^2 + \epsilon/2$
 $\leq \theta(a * a) + \epsilon/2 \leq \bar{\theta}(v * v) + \epsilon \leq 1 + \epsilon$. Since ϵ is arbitrary,
 f is an element of V^0 . Conversely, if g is an element
of V^0 and a is any element of U , then $|g(\tau a)| \leq 1$
since τa is in V . Therefore, $g \circ \tau$ is in U^0 and $\overline{g \circ \tau} = g$.

Now let $\{f_\alpha\}$ be a net in U^0 converging to 0 in
the $\sigma(\langle P \rangle, A)$ topology. For each c in $C(A)$ and $\epsilon > 0$
there exists an element a in A such that $|c - \tau a| < \epsilon/2 |\bar{\theta}|^{\frac{1}{2}}$,
and there is α_0 such that $|f_\alpha(a)| < \epsilon/2$ whenever $\alpha > \alpha_0$.
Since each \bar{f}_α is an element of V^0 and so $|\bar{f}_\alpha(x)| \leq \bar{\theta}(x * x)^{\frac{1}{2}}$
for all x in $C(A)$, we see that $|\bar{f}_\alpha| \leq |\bar{\theta}|^{\frac{1}{2}}$. Thus
 $|\bar{f}_\alpha(c)| < \epsilon/2 + |\bar{f}_\alpha(\tau a)| < \epsilon$ whenever $\alpha > \alpha_0$. The
 $\sigma(C(A)^*, C(A)) - \sigma(\langle P \rangle, A)$ continuity of the inverse of the
mapping $f \rightarrow \bar{f}$ is clear.

4.4.4. Lemma. For each F in $(A, \mathscr{P}^1)^\wedge$ there exists a
unique element \bar{F} in $C(A)^{**}$ such that $\bar{F}(g) = F(G \circ \tau)$ for all
 g in $C(A)^*$.

Proof. For F in $(A, \mathscr{P}^1)^\wedge$ define \bar{F} as a linear

functional on $C(A)^*$ by $\bar{F}(g) = F(f \circ \tau)$. Since the mapping $g \rightarrow g \circ \tau$ is a bijection from $C(A)^*$ to $\langle P \rangle$ by Lemma 4.4.2, it is clear that \bar{F} is uniquely defined. To show that \bar{F} is an element of $C(A)^{**}$, it suffices to show that if $\{g_n\}$ is a sequence in $C(A)^*$ such that $\lim |g_n| = 0$, then the sequence $\{\bar{F}(g_n)\}$ clusters at 0. By choosing a subsequence if necessary, we may assume that $|g_n| \leq 1/2^n$.

For each n there exist $\varphi_{i,n}$ in P' and $\lambda_{i,n}$ complex numbers ($i = 1, 2, 3, 4$) by [7, p.40] such that $g_n = \sum_{i=1}^4 \lambda_{i,n} \varphi_{i,n}$, where $|\varphi_{1,n}| + |\varphi_{2,n}| = |\varphi_{1,n} - \varphi_{2,n}| = \frac{1}{2} |g_n + g_n^*| \leq |g_n|$ and $|\varphi_{3,n}| + |\varphi_{4,n}| = |\varphi_{3,n} - \varphi_{4,n}| = \frac{1}{2} |g_n - g_n^*| \leq |g_n|$. Then $\varphi_i = \sum_{n=1}^{\infty} \varphi_{i,n}$ for $i = 1, 2, 3, 4$, is a well-defined element of $C(A)^*$, and $\varphi_i \circ \tau = \sum_{n=1}^{\infty} \varphi_{i,n} \circ \tau$ is a well-defined element of P in A^* by Lemma 4.4.3. Moreover, for each a in A we have that $|\varphi_{i,n} \circ \tau(a)|^2 \leq \|\varphi_{i,n} \circ \tau\| \varphi_{i,n}(\tau a^* \tau a) \leq |\varphi_{i,n}| \varphi_i(\tau a^* \tau a) \leq \varphi_i \circ \tau(a^* a)$ for each i and n [7, p.23]. Therefore, $\{\varphi_{i,n} \circ \tau\}$ is a sequence for fixed i in the polar of the \mathcal{J}^1 -neighborhood $\{a \in A : \varphi_i \circ \tau(a^* a) \leq 1\}$; moreover, $\lim \|\varphi_{i,n} \circ \tau\| = \lim |\varphi_{i,n}| = 0$. Therefore, $\lim \bar{F}(g_n) = \lim \bar{F}(\sum_{i=1}^4 \lambda_{i,n} \varphi_{i,n}) = \lim \sum_{i=1}^4 \lambda_{i,n} \bar{F}(\varphi_{i,n}) = \lim \sum_{i=1}^4 \lambda_{i,n} F(\varphi_{i,n} \circ \tau) = 0$ since F is in $(A, \mathcal{J}^1)^\wedge$.

4.4.5 Theorem. The \mathcal{J} -completion of A and \mathcal{J}^1 -completion of A are the same subsets of $\langle P \rangle'$.

Proof. Since the collection of \mathcal{J}^1 -neighborhoods of the

form $\{a \in A: \theta(a*a) \leq 1\}$ and of \mathcal{A}^2 -neighborhoods of the form $\{a \in A: \theta(aa^*) \leq 1\}$ for θ in P is a subbase for \mathcal{A} , it follows directly that $(A, \mathcal{A})^\wedge$ is a subset of $(A, \mathcal{A}^1)^\wedge$. Furthermore, since this subbase for \mathcal{A} satisfies the hypotheses of Corollary 4.1.3, to show the reverse containment, it suffices by that corollary to prove that each element of $(A, \mathcal{A}^1)^\wedge$ is $\sigma(\langle P \rangle, A)$ -continuous on the polar of each \mathcal{A}^2 -neighborhood of the form $\{a \in A: \theta(aa^*) \leq 1\}$ for θ in P . It is enough, therefore, to show that if F is any element of $(A, \mathcal{A}^1)^\wedge$ and $\{f_\alpha\}$ is any net in $U^0 = \{a \in A: \theta(aa^*) \leq 1\}^0$ which converges to 0 in the $\sigma(\langle P \rangle, A)$ topology, then $\{F(f_\alpha)\}$ clusters at 0 .

Let \bar{F} , $\bar{\theta}$, and \bar{f} for each α denote the elements of $C(A)^{**}$ and $C(A)^*$ defined as in Lemmas 4.4.4 and 4.4.2. It is then easy to check that $F(f_\alpha) = \bar{F}(\bar{f}_\alpha)$ for each α , and we shall prove that the net $\{\bar{F}(\bar{f}_\alpha)\}$ clusters at 0 .

The net $\{\bar{f}_\alpha\}$ is contained in the polar of the set $V = \{c \in C(A): \bar{\theta}(cc^*) \leq 1\}$ by Lemma 4.4.3; moreover, V^0 is $\sigma(C(A)^*, C(A)^{**})$ -compact: Corollary 3.2.9 asserts that the polar of the set $W = \{c \in C(A): \bar{\theta}(c^*c) \leq 1\}$ is $\sigma(C(A)^*, C(A)^{**})$ -compact, and V^0 is the image of W^0 under the $\sigma(C(A)^*, C(A)^{**})$ -continuous mapping $g \rightarrow g^*$ on $C(A)^*$.

Choose g in $C(A)^*$ to be a $\sigma(C(A)^*, C(A)^{**})$ -cluster point of the net $\{\bar{f}_\alpha\}$; then, $\bar{F}(g)$ is a cluster point of the net $\{\bar{F}(\bar{f}_\alpha)\}$. Since the net $\{\bar{f}_\alpha\}$ converges to 0 in

the $\sigma(C(A)^*, C(A))$ topology by Lemma 4.4.3, we see that g must be 0, Hence $\{\bar{F}(\bar{f}_\alpha)\}$ clusters at 0 as was to be shown. The proof is thereby concluded.

We now show that there is a natural involution on $(A, \mathcal{A})^\wedge$ which extends the involution on A .

4.4.6 Lemma. For f in $\langle P \rangle$ the element f^* , defined in A^* , is also in $\langle P \rangle$. Then if F is any element of $(A, \mathcal{A})^\wedge$, the linear functional F^* on $\langle P \rangle$ defined by $F^*(f) = \overline{F(f^*)}$ is an element of $(A, \mathcal{A})^\wedge$.

Proof. Since f in $\langle P \rangle$ can be written as $f = \sum_{i=1}^4 \lambda_i \theta_i$ and $\theta_i^*(x) = \theta_i(x)$ for each x in A by [7, p.23], we see that $f^* = \sum_{i=1}^4 \bar{\lambda}_i \theta_i$ is an element of $\langle P \rangle$. To show that F^* is an element of $(A, \mathcal{A})^\wedge$, it suffices by Theorem 4.4.5 to prove that F^* is $\sigma(\langle P \rangle, A)$ -continuous on the polar of each \mathcal{A}^1 -neighborhood of the form $\{a \in A : \theta(a^*a) \leq 1\}$. However, a net $\{f_\alpha\}$ in the polar of the \mathcal{A}^1 -neighborhood $U = \{a \in A : \theta(a^*a) \leq 1\}$ for θ in P converges to 0 in the $\sigma(\langle P \rangle, A)$ topology if and only if the net $\{f_\alpha^*\}$ in the polar of the \mathcal{A}^2 -neighborhood $U^* = \{a \in A : \theta(aa^*) \leq 1\}$ converges to 0 in this same topology. Since F is $\sigma(\langle P \rangle, A)$ -continuous on both U^0 and $(U^*)^0$, the lemma now follows.

4.4.7 Corollary. The \mathcal{A} -completion of A and the \mathcal{A}^2 -completion of A are the same subset of $\langle P \rangle'$.

Proof. Since F is in $(A, \mathcal{A}^2)^\wedge$ if and only if F^* is in $(A, \mathcal{A}^1)^\wedge$ as we see from the proof above, the corollary follows from Theorem 4.4.5 and Lemma 4.4.6.

We have now shown that $(A, \mathcal{A}^1)^\wedge$, $(A, \mathcal{A}^2)^\wedge$, and $(A, \mathcal{A})^\wedge$ are the same.

4.4.8 Theorem. The \mathcal{A} -completion of A is a $*$ -algebra under either Arens multiplication, and the Arens multiplications are regular on $(A, \mathcal{A})^\wedge$.

Proof. We first show that either Arens multiplication can be defined on $(A, \mathcal{A})^\wedge$ by verifying the two conditions of Definition 3.1.1. Since multiplication on A is separately \mathcal{A} -continuous by Theorem 2.1.4, we have that $(A, \mathcal{A})^*$ is a left and a right A -module. Secondly, if F is in $(A, \mathcal{A})^\wedge$ and f is in $(A, \mathcal{A})^*$, then $F \cdot f$ and $F:f$ are defined as elements of A' ; and it is easy to check that $F \cdot f(a) = \bar{F} \cdot \bar{f}(\tau a)$ and $F:f(a) = \bar{F}:\bar{f}(\tau a)$ for each a in A , where \bar{F} in $C(A)^{**}$ and \bar{f} in $C(A)^*$ are defined as in Lemma 4.4.4 and 4.4.2 respectively. Since $\bar{F} \cdot \bar{f}$ and $\bar{F}:\bar{f}$ are elements of $C(A)^*$, we have that $F \cdot f = (\bar{F} \cdot \bar{f}) \circ \tau$ and $F:f = (\bar{F}:\bar{f}) \circ \tau$ are elements of $\langle P \rangle$ by Lemma 4.4.2. Thus, $(A, \mathcal{A})^* = \langle P \rangle$ is invariant under the mappings $(F, f) \rightarrow F \cdot f$ and $(F, f) \rightarrow F:f$ from $(A, \mathcal{A})^\wedge \times (A, \mathcal{A})^*$ to $(A, \mathcal{A})'$.

Next, to show that $(A, \mathcal{A})^\wedge$ is an algebra under the multiplication $(F, G) \rightarrow FG$, it suffices by Lemma 3.1.2 and

the bilinearity of the mapping $(F, G) \rightarrow FG$ from $(A, \mathcal{A})^\wedge \times (A, \mathcal{A})^\wedge$ to $(A, \mathcal{A})^{**}$ to prove that each FG is actually an element of $(A, \mathcal{A})^\wedge$. In view of Corollary 4.4.7 it suffices to show that FG is $\sigma(\langle P \rangle, A)$ -continuous on the polar of each \mathcal{A}^2 -neighborhood of the form $U = \{a \in A : \theta(aa^*) \leq 1\}$ for θ in P .

With the notation as in Lemmas 4.4.2 and 4.4.4 we need to establish the following relations: for each f in $\langle P \rangle$ and a and x in A we have that $f \cdot a(x) = f(ax) = \bar{f}(\tau(ax)) = \bar{f} \cdot \tau a(\tau x) = (\bar{f} \cdot \tau a) \circ \tau(x)$; therefore $G \cdot f(a) = G(f \cdot a) = G(\bar{f} \cdot \tau a) \circ \tau = \bar{G}(\bar{f} \cdot \tau a) = ((\bar{G} \cdot \bar{f}) \circ \tau)(a)$ and $FG(f) = F(G \cdot f) = F((\bar{G} \cdot \bar{f}) \circ \tau) = \bar{F}(\bar{G} \cdot \bar{f}) = \bar{F}\bar{G}(\bar{f})$. Then if $\{f_\alpha\}$ is any net in $U^0 = \{a \in A : \theta(aa^*) \leq 1\}^0$ converging to 0 in the $\sigma(\langle P \rangle, A)$ topology the net $\{\bar{f}_\alpha\}$ is contained in the polar (in $C(A)^*$) of the set $V = \{c \in C(A) : \bar{\theta}(cc^*) \leq 1\}$ and $\{\bar{f}_\alpha\}$ converges to 0 in the $\sigma(C(A)^*, C(A))$ topology by Lemma 4.4.3. Since the set V^0 is $\sigma(C(A)^*, C(A)^{**})$ -compact as a consequence of Corollary 3.2.9, the net $\{\bar{f}_\alpha\}$ must converge to 0 in the $\sigma(C(A)^*, C(A)^{**})$ topology; thus, $\lim FG(f_\alpha) = \lim \bar{F}\bar{G}(\bar{f}_\alpha) = 0$.

It is easy to verify that $(F^*G^*)^* = F:G$ and $F:G(f) = \bar{F}:\bar{G}(\bar{f})$ for F and G in $(A, \mathcal{A})^\wedge$ and f in $\langle P \rangle$. Since the Arens multiplications are regular on $C(A)^{**}$ by Theorem 3.2.5, we see that $FG(f) = \bar{F}\bar{G}(\bar{f}) = \bar{G}:\bar{F}(\bar{f}) = G:F(f)$ for every F and G in $(A, \mathcal{A})^\wedge$ and f in $\langle P \rangle$.

Thus, $(A, \mathscr{A})^\wedge$ is a $*$ -algebra under either Arens multiplication and these multiplications are regular.

The completion of A with respect to \mathscr{A} under either Arens multiplication is essentially the only example of a locally convex completion which we can prove is an algebra under some Arens multiplication without being able to verify the hypotheses of Theorem 4.1.5. However, Theorem 4.1.5 asserts not only that certain locally convex completions are algebras under an Arens multiplication but also that the multiplication is continuous in the left variable with respect to the completion topology.

The mappings $G \rightarrow \max\{GG^*(\varphi)^{\frac{1}{2}}, G^*G(\varphi)^{\frac{1}{2}}\}$ on $C(A)^{**}$ for φ in P' are seminorms since $C(A)$ is a B^* -algebra, and the collection of these seminorms defines a locally convex Hausdorff topology on $C(A)^{**}$ called \mathscr{A}_e (see Definition 3.2.6). In fact, $\sigma(C(A)^{**}, C(A)^*) \leq \mathscr{A}_e \leq \tau(C(A)^{**}, C(A)^*)$ by Theorems 3.2.15 and 3.2.10.

For the following theorem we are considering $(A, \mathscr{A})^\wedge$ and $C(A)^{**}$ as algebras under the Arens multiplication $(F, G) \rightarrow FG$.

4.4.9 Theorem. There exists a $*$ -isomorphism $\Gamma: (A, \mathscr{A})^\wedge \rightarrow C(A)^{**}$ which is a homeomorphism with respect to the \mathscr{A} -completion topology on $(A, \mathscr{A})^\wedge$ and the \mathscr{A}_e topology on $C(A)^{**}$. Moreover, $\Gamma a = a$ for every a in A .

Proof. For F in $(A, \mathcal{A})^\wedge$ define ΓF by Lemma 4.4.4: that is, as the unique element \bar{F} in $C(A)^{**}$ such that $\bar{F}(g) = F(g \circ \tau)$ for all g in $C(A)^*$. It is now clear that Γ is well-defined, linear and $\Gamma a = a$ for every a in A . Moreover, if $\Gamma F = \Gamma G$ for F and G in $(A, \mathcal{A})^\wedge$, then $F(f) = \bar{F}(\bar{f}) = \bar{G}(\bar{f}) = G(f)$ for each f in $(A, \mathcal{A})^*$, where \bar{f} is defined as in Lemma 4.4.2; therefore, Γ is injective. Furthermore, we showed in the proof of Theorem 4.4.8 that $FG(f) = \bar{F}\bar{G}(\bar{f})$ for each F and G in $(A, \mathcal{A})^\wedge$ and f in $(A, \mathcal{A})^*$; thus, Γ is an algebra homomorphism.

If G is an element of $C(A)^{**}$, define a linear functional F on $(A, \mathcal{A})^*$ by $F(f) = G(\bar{f})$. Lemma 4.4.3 and Corollary 3.2.9 which asserts the $\sigma(C(A)^*, C(A)^{**})$ -compactness of each of the sets $V^\theta = \{c \in C(A) : \bar{\theta}(c^*c) \leq 1\}^\circ$ for each θ in P ensures us that F is an element of $(A, \mathcal{A}^2)^\wedge$ and, hence, of $(A, \mathcal{A})^\wedge$ by Corollary 4.4.7. Since it is easily verified that $\Gamma(F^*) = (\Gamma F)^*$, we see that Γ is a $*$ -isomorphism.

The collection of sets of the form $U_\theta = \{a \in A : \theta(a^*a) \leq 1\}$ and $(U^*)_\theta = \{a \in A : \theta(aa^*) \leq 1\}$ for θ in P constitute a subbase for the \mathcal{A} topology on A ; Therefore, we have that the collection of sets $U_\theta^\infty \cap (A, \mathcal{A})^\wedge$ and $(U^*)_\theta^\infty \cap (A, \mathcal{A})^\wedge$ (where the bipolars are taken in $(A, \mathcal{A})^{*'}$) form a subbase for the \mathcal{A} -completion topology by Corollary 4.1.3. Let V_θ denote the set $\{c \in C(A) : \bar{\theta}(c^*c) \leq 1\}$; then $V_\theta^\circ = \{\bar{f} \in C(A)^{**} : f \in U_\theta^\circ\}$ by

Lemma 4.4.3. It is easy to check that $\Gamma(U_\theta^{\circ\circ} \cap (A, \mathscr{A})^\wedge)$ equals $V_\theta^{\circ\circ}$ in $C(A)^{**}$; moreover, $V_\theta^{\circ\circ}$ equals the set $\{G \in C(A)^{**} : G * G(\bar{\theta}) \leq 1\}$ by Lemma 3.2.8. Since $(U^*)_\theta$ is the set $\{u \in A : u^* \in U_\theta\}$, and if $(V^*)_\theta$ is defined to be the set $\{v \in C(A) : v^* \in V_\theta\}$, then $\Gamma((U^*)_\theta^{\circ\circ} \cap (A, \mathscr{A})^\wedge) = (V^*)_\theta^{\circ\circ} = \{G \in C(A)^{**} : GG^*(\bar{\theta}) \leq 1\}$ for the same reasons. Since the mapping $\theta \mapsto \bar{\theta}$ is a bijection from P to P' by Lemma 4.4.2, we see that Γ is also a homeomorphism with respect to the \mathscr{A} -completion topology on $(A, \mathscr{A})^\wedge$ and the \mathscr{A}_e topology on $C(A)^{**}$.

The proof of the following corollary is immediate.

4.4.10 Corollary. If A is a B^* -algebra, then $(A, \mathscr{A})^\wedge = A^{**}$.

CHAPTER V

THE \mathcal{J}_e AND μ_e TOPOLOGIES ON W^* -ALGEBRAS

In Chapter III we studied the topologies \mathcal{J}_e and μ_e on the biduals of certain Banach algebras. An alternative problem is to consider a W^* -algebra M -- that is, a B^* -algebra which is the dual of a Banach space F -- with the topologies induced on M by the same seminorms as generate \mathcal{J}_e and μ_e .

The \mathcal{J}_e topology and related topologies on W^* -algebras have been widely investigated by Sakai [20] and others. Akemann, for instance, has F of M to be relatively $\sigma(F, M)$ -compact and, thereby, proves that the \mathcal{J}_e and $\tau(M, F)$ topologies agree on norm balls of M [1]. We show that for certain types of Banach algebras A a subset of A^* satisfies Akemann's criterion if and only if it is equicontinuous with respect to a topology on A related to the \mathcal{J} topology (see Theorem 5.1.6). We then investigate this criterion in terms of the duals and biduals of AB^* -algebras. We also make some remarks about the μ_e topology on W^* -algebras: this topology was first introduced by P. C. Shields in [24], but was discovered independently by this author.

The first section establishes terminology and includes

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additional relations on the topologies of the β , μ , and \mathcal{A} families.

Section 1. Additional Relations on the Topologies of the β , μ , and \mathcal{A} Families.

The reader is referred to [29] for the following definition and proof of the first lemma.

5.1.1. Definition. If A is a Banach space and \mathfrak{T} is a locally convex Hausdorff topology on A such that $\mathfrak{T} \leq \tau(A, A^*)$, then there exists a finest locally convex topology on A which agrees with \mathfrak{T} on norm-balls of A . Moreover, if we denote this topology by \mathfrak{T}^{lf} , then $\mathfrak{T} \leq \mathfrak{T}^{lf} \leq \tau(A, A^*)$. A base of neighborhoods for \mathfrak{T}^{lf} consists of sets of the form $V = \bigcup_{n=1}^{\infty} (U_1 \cap S + U_2 \cap 2S + \dots + U_n \cap nS)$, where S denotes the unit ball of A and $\{U_i\}$ is a sequence of \mathfrak{T} -neighborhoods.

5.1.2 Lemma. Let \mathfrak{T} be a locally convex Hausdorff topology on a Banach space A such that $\mathfrak{T} \leq \tau(A, A^*)$. Then we have the following: (i) if \mathfrak{T}' is a locally convex Hausdorff topology on A , then $\mathfrak{T}' \leq \mathfrak{T}^{lf}$ if $\mathfrak{T}'|_S \leq \mathfrak{T}|_S$; and (ii) a linear functional f on A is \mathfrak{T}^{lf} continuous if and only if $f|_S$ is \mathfrak{T} -continuous.

F. D. Santilles has proved the next two theorems for

the β^1 topology in an A-module setting [21]; and D. C. Taylor, using some of the same techniques, has proved them for the β topology on a B*-algebra [26]. We restrict our attention to a Banach algebra A with bounded approximate identity and give modified versions of these proofs for the β topology on A.

5.1.3 Theorem. Let W be an absolutely convex subset of a Banach algebra A with bounded approximate identity, and suppose that for each positive integer n there is a β -neighborhood V_n such that $V_n \cap nS$ is contained in W. Then the β -closure of W is a β -neighborhood.

Proof. By hypothesis and the remarks following Definition 2.1.1 choose a sequence $\{V_n\}$ of β -neighborhoods such that each V_n is of the form $\{x \in A : \|a_n x\| \leq 1, \|x b_n\| \leq 1\}$ for a_n and b_n in A and such that $V_n \cap nS$ is a subset of W. Then $\bigcup_{n=1}^{\infty} (V_n \cap nS)$ is contained in W.

We now show that $\lim \|f \cdot e_{\alpha} + e_{\alpha} \cdot f - e_{\alpha} \cdot f \cdot e_{\alpha} - f\| = 0$ uniformly on f in W^0 (taken in A^*), where $\{e_{\alpha}\}$ is a bounded approximate identity for A. Let $\epsilon > 0$ and choose a positive integer N such that $1/n < \epsilon/4$. Then there is α_0 such that $\alpha > \alpha_0$ implies $\max\{\|a_N e_{\alpha} - a_N\|, \|e_{\alpha} b_N - b_N\|\} < 1/N$. For x in A of norm at most 1 the set $\{(N/4)(e_{\alpha} x + x e_{\alpha} - e_{\alpha} x e_{\alpha} - x) : \alpha > \alpha_0\}$ is contained in $V_N \cap NS$, a subset of W. Hence, for every f in W^0 we have that

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$|f(e_\alpha x + x e_\alpha - e_\alpha x e_\alpha - x)| < 4/N < \epsilon$ whenever $\alpha > \alpha_0$. Therefore, $\|f \cdot e_\alpha + e_\alpha \cdot f - e_\alpha \cdot f \cdot e_\alpha - f\| < \epsilon$ for each f in W^0 whenever $\alpha > \alpha_0$.

Furthermore, W^0 is norm-bounded in A^* . To see this, note that the set $(1 + \|a_N\| + \|b_N\|)^{-1} \cdot S$ is contained in $V_N \cap NS$; hence, W^0 is a subset of $(1 + \|a_N\| + \|b_N\|) \cdot \{f \in A^* : \|f\| \leq 1\}$. In Theorem 2.1.7 we gave a proof that the set $X = \{f \in A^* : \lim \|e_\alpha \cdot f + f \cdot e_\alpha - e_\alpha \cdot f \cdot e_\alpha - f\| = 0\}$ is contained in $(A, \beta)^*$; the same proof can be easily adapted to show that W^0 , a subset of X , is contained in the polar of some β -neighborhood. Therefore, W^{00} in A is a β -neighborhood; moreover, W^{00} is the β -closure of W since W is absolutely convex by [18, p.36]. This concludes the proof.

5.1.4 Theorem. If A is a Banach algebra with bounded approximate identity, then the topologies β and β^{lf} are the same.

Proof. From Definition 5.1.1 we have that $\beta \leq \beta^{lf}$. To show the reverse relationship, we choose a base \mathcal{V} for β^{lf} -neighborhoods consisting of β^{lf} -closed absolutely convex neighborhoods by [18, p.12]. We see from Definition 5.1.1 that each V in \mathcal{V} satisfies the hypotheses of Theorem 5.1.3. Therefore, if we show that β^{lf} -closed convex sets are β -closed, the desired conclusion follows from Theorem 5.1.3. It thus suffices to prove that

$(A, \beta^{1f})^* = (A, \beta)^*$: for then, the β -closed convex sets and the β^{1f} -closed convex sets are the same by [18, p.34].

To that end, note that $(A, \beta^{1f})^*$ contains $(A, \beta)^*$ and is a subset of A^* by Lemma 5.1.2. If f is any element of $(A, \beta^{1f})^*$, then f is β -continuous by Theorem 2.1.7 provided that $\lim \|f \cdot e_\alpha + e_\alpha \cdot f - e_\alpha \cdot f \cdot e_\alpha - f\| = 0$, where $\{e_\alpha\}$ is a bounded approximate identity for A . Let $\epsilon > 0$; then since $f|_S$ is β -continuous by Lemma 5.1.2, there exists a β -neighborhood $V = \{x \in A : \|ax\| \leq 1, \|xb\| \leq 1\}$ for some a and b in A such that $|f(v)| < \epsilon$ for every v in V . Choose α_0 so that $\max \{\|ae_\alpha - a\|, \|e_\alpha b - b\|\} < \frac{1}{2}$ whenever $\alpha > \alpha_0$. Consequently, if x is any element of A with norm at most 1, the set $\{e_\alpha x + xe_\alpha - e_\alpha x e_\alpha - x : \alpha > \alpha_0\}$ is contained in V and so $|f(e_\alpha x + xe_\alpha - e_\alpha x e_\alpha - x)| < \epsilon$ for $\alpha > \alpha_0$. The proof is now completed.

It is not necessarily the case that β and β^{1f} are equivalent (see Example 5.1.8).

5.1.5 Lemma. Let A be a Banach $*$ -algebra with bounded approximate identity such that the set $P = \{\theta \in A^* : \theta(x^*x) \geq 0, \forall x \in A\}$ separates points of A . For θ in P let $U = \{x \in A : \theta(x^*x) \leq 1, \theta(xx^*) \leq 1\}$, let $V' = \{F \in A^{**} : F^*F(\theta) \leq 1, FF^*(\theta) \leq 1\}$, and let N be a positive integer. If S and S' denote the unit balls of A and A^{**} respectively, then $V' \cap NS'$ is contained in $(U \cap NS)^{oo}$,

where the bipolar is taken in A^{**} .

Proof. Let F be an element of $V' \cap NS'$ and let f be any element of $(U \cap NS)^{\circ}$ in A^* . For $0 < \epsilon < 1$ we can choose an element a in A by Lemma 3.2.13 and Goldstine's Theorem which satisfies the following properties: (i) $\|a\| \leq \|(1 - \epsilon)F\| \leq N$; (ii) $\max \{ |((1 - \epsilon)^2 F^*F - a^*a)(\theta)|, |((1 - \epsilon)^2 FF^* - aa^*)(\theta)| \} < \epsilon$; and (iii) $|((1 - \epsilon)F - a)(f)| < \epsilon$. Then $\max\{\theta(a^*a), \theta(aa^*)\} < \epsilon + (1 - \epsilon)^2 \cdot \max\{F^*F(\theta), FF^*(\theta)\} < \epsilon + (1 - \epsilon)^2 \leq 1$; in particular, a is an element of $U \cap NS$. Hence, $(1 - \epsilon)|F(f)| < \epsilon + |f(a)| \leq \epsilon + 1$ since f is an element of $(U \cap NS)^{\circ}$. It now follows that F is an element of $(U \cap NS)^{\circ\circ}$.

If M is a W^* -algebra with predual F , Akemann [1] has given several necessary and sufficient conditions for a (norm-bounded) subset K of F to be relatively $\sigma(F, M)$ -compact, one of which is the following: there exists an element θ in F such that $\theta(x^*x) \geq 0$ for all x in M and which has the property that given $\epsilon > 0$ there is $\delta > 0$ such that whenever x is in M with $\|x\| \leq 1$ and $\theta(x^*x + xx^*) < \delta$, then $|f(x)| < \epsilon$ for all f in K . We now prove that for a certain type of Banach algebra A a subset K of A^* satisfies Akemann's criterion if and only if K is \mathcal{L}^1 -equicontinuous.

5.1.6 Theorem. Let A be a Banach $*$ -algebra with bounded

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approximate identity such that P separates points of A . If K is a subset of A^* , then the following are equivalent: (i) there exists an \mathscr{A}^{lf} -neighborhood V in A such that K is contained in V^0 in A^* ; (ii) there exists θ in P with $\|\theta\| \leq 1$ such that given $\epsilon > 0$ there is $\delta > 0$ such that if x is in A with $\|x\| \leq 1$ and $\max \{\theta(x*x), \theta(xx^*)\} < \delta$, then $|f(x)| < \epsilon$ for all f in K ; and (iii) there exists θ in P with $\|\theta\| \leq 1$ such that given $\epsilon > 0$ there is $\delta > 0$ such that if F is in A^{**} with $\|F\| \leq 1$ and $\max \{F*F(\theta), FF^*(\theta)\} < \delta$, then $|F(f)| < \epsilon$ for all f in K .

Proof of (i) \Rightarrow (iii). By Definition 5.1.1. the \mathscr{A}^{lf} -neighborhood V contains a set of the form $\bigcup_{n=1}^{\infty} (U_1 \cap S + \dots + U_n \cap nS)$, where S is the unit ball of A and $\{U_n\}$ is a sequence of \mathscr{A} -neighborhoods. Thus, for each positive integer n there exists an \mathscr{A} -neighborhood U_n such that $\sup \{|f(x)| : f \in K\} \leq 1$ for all x in $U_n \cap nS$. Moreover, each \mathscr{A} -neighborhood U_n contains a set of the form $V_n = \{x \in A : \theta_n(x*x) \leq 1, \theta_n(xx^*) \leq 1\}$ by the remarks following Definition 2.1.3.

Let $\theta = \sum_{n=1}^{\infty} (1/(2^n \cdot (\|\theta_n\| + 1))) \theta_n$; then θ is a well-defined element of P such that $\|\theta\| \leq 1$. Given $\epsilon > 0$, choose a positive integer N such that $1/N < \epsilon$ and set δ equal to $1/(N^2 \cdot 2^n \cdot (\|\theta_n\| + 1))$. If F is an element of A^{**} with $\|F\| \leq 1$ and $\max \{F*F(\theta), FF^*(\theta)\} < \delta$, then NF is an

element of $(V_N \cap NS)^{\circ\circ}$ by Lemma 5.1.5: in fact,
 $\max \{N^2 F^* F(\theta_N), N^2 FF^*(\theta_n)\} \leq N^2 \cdot 2^N \cdot (\|\theta_N\| + 1) \max F^* F(\theta), FF^*(\theta) \leq 1$.
 Since $V_N \cap NS$ is contained in $U_N \cap NS$, a subset of V , we have that K is contained in $(V_N \cap NS)^{\circ}$. Therefore,
 $\sup \{N|F(f)| : f \in K\} \leq 1$; that is, $\sup \{|F(f)| : f \in K\} \leq 1/N < \epsilon$.

Clearly, (iii) \Rightarrow (ii). It remains to show that (ii) \Rightarrow (i). For each positive integer n let δ_n be chosen that $\sup \{|f(x)| : f \in K\} < 1/2^n \cdot n$ whenever x is in A with $\|x\| \leq 1$ and $\max \{\theta(x*x), \theta(xx^*)\} < \delta_n$. Let $\{U_n\}$ be the sequence of \mathcal{A} -neighborhoods defined by $U_n = \{x \in A : \theta(x*x) \leq \delta_n, \theta(xx^*) \leq \delta_n\}$; then, $V = \bigcup_{n=1}^{\infty} (U_1 \cap S + \dots + U_n \cap nS)$ is an \mathcal{A}^{lf} -neighborhood. Let $x = x_1 + x_2 + \dots + x_N$ be an element of V , where x_n is in $U_n \cap nS$ for $n = 1, \dots, N$. Then $\|(1/n)x_n\| \leq 1$ and $\max \{\theta((1/n^2)x_n*x_n), \theta((1/n^2)x_n x_n^*)\} \leq \delta_n/n^2 \leq \delta_n$; thus, $|f(x)| \leq \sum_{n=1}^N |f(x_n)| \leq \sum_{n=1}^N n |f((1/n)x_n)| \leq \sum_{n=1}^N 1/2^n \leq 1$ for each f in K . Therefore, K is contained in V° and the proof is completed.

5.1.7 Corollary. If A is a Banach $*$ -algebra with bounded approximate identity such that P separates points of A , then $(A, \mathcal{A}^{lf})^*$ is the norm-closure of $(A, \mathcal{A})^*$ in A^* .

Proof. From Definition 5.1.1 we have that $\mathcal{A} \leq \mathcal{A}^{lf} \leq \tau(A, A^*)$; therefore, $(A, \mathcal{A})^*$ is contained in

$(A, \mathscr{A}^{lf})^*$, which is a subset of A^* . Furthermore, it is easy to show that $(A, \mathscr{A}^{lf})^*$ is norm-closed in A^* .

Suppose there exists an element f of $(A, \mathscr{A}^{lf})^*$ which is not in the norm-closure of $(A, \mathscr{A})^*$. Then there exists by the Hahn-Banach Theorem an element F of A^{**} such that $F(f) > 0$ and $f(g) = 0$ for all g in $(A, \mathscr{A})^*$, and we may assume that $\|F\| = 1$. Since $(A, \mathscr{A})^*$ equals $\langle P \rangle$ by Theorem 2.1.8, we have that $F^*F(\theta) = 0 = FF^*(\theta)$ for every θ in P by Lemma 3.2.12. Choosing ϵ such that $0 < \epsilon < F(f)$ in part (iii) of the theorem applied to f , we arrive at a contradiction. Consequently, $(A, \mathscr{A}^{lf})^*$ is the norm-closure of $(A, \mathscr{A})^*$.

The preceding theorem and corollary can be adapted to show that $(A, (\mathscr{A}_i)_{lf})^* = \text{norm-cl}(A, \mathscr{A}_i)^*$ for $i = 1, 2$.

5.1.8 Example. We now see that \mathscr{A} and \mathscr{A}^{lf} are not necessarily the same topologies. If A is an AB^* -algebra which is not a B^* -algebra under any equivalent norm, as Example 3.3.7, then $(A, \mathscr{A}^{lf})^* = A^*$ by the Theorems 3.2.14, 2.1.8, and Corollary 5.1.7; but $(A, \mathscr{A})^*$ cannot be A^* in view of Corollary 2.1.9.

We have not investigated these types of relationships for the μ family, but we do generalize Theorem 2.2.3 as follows.

5.1.9 Theorem. If A is an AB^* -algebra with bounded approximate identity, then (i) $\sigma(A, A^*) \leq \mu^1 \leq (\mathscr{J}^2)^{1f} \leq \tau(A, A^*)$; (ii) $\sigma(A, A^*) \leq \mu^2 \leq (\mathscr{J}^1)^{1f} \leq \tau(A, A^*)$; and (iii) $\sigma(A, A^*) \leq \mu \leq \mathscr{J}^{1f} \leq \tau(A, A^*)$.

Proof. We shall prove (i); then (ii) follows analogously, and (iii) is an immediate consequence of (i) and (ii). It suffices to show that $\mu^1 \leq (\mathscr{J}^2)^{1f}$ by Theorem 2.1.4 and Definition 5.1.1. Let $\{a_\alpha\}$ be a net in the unit ball of A which converges to 0 in the \mathscr{J}^2 topology and let $\epsilon > 0$. Then since A is AB^* , for each f in A^* there exists θ_i in $P = \{\theta \in A^* : \theta(x^*x) \geq 0, \forall x \in A\}$ and complex numbers $\lambda_i (i=1,2,3,4)$ such that $\|f - \sum_{i=1}^4 \lambda_i \theta_i\| < \epsilon/2$ by Theorem 3.2.14. There exists α_0 such that $\alpha > \alpha_0$ implies $\max\{\theta_i(a_\alpha a_\alpha^*)^{\frac{1}{2}} : i=1,2,3,4\} < \epsilon/2 \cdot (1 + \sum_{i=1}^4 |\lambda_i| \|\theta_i\|^{\frac{1}{2}})$. Then since $\|a_\alpha x\| \leq 1$ for each x in S and each α , we have by the Cauchy-Schwartz inequality that $|f(a_\alpha x)| < \epsilon/2 + \sum_{i=1}^4 |\lambda_i| |\theta_i(a_\alpha x)| \leq \epsilon/2 + \sum_{i=1}^4 |\lambda_i| \theta_i(a_\alpha a_\alpha^*)^{\frac{1}{2}} \cdot \theta_i(x^*x)^{\frac{1}{2}} \leq \epsilon/2 + \sum_{i=1}^4 |\lambda_i| \|\theta_i\|^{\frac{1}{2}} \theta(a_\alpha a_\alpha^*)^{\frac{1}{2}}$; therefore, $\|f \cdot a_\alpha\| < \epsilon$ whenever $\alpha > \alpha_0$. Since each norm-bounded \mathscr{J}^2 -convergent net converges in the μ^1 topology, by Lemma 5.1.2 we have that $\mu^1 \leq (\mathscr{J}^2)^{1f}$.

5.1.10 Theorem. Let A be an AB^* -algebra with bounded approximate identity. If S denotes the unit ball of A^{**} , then $\mu_e|_S \leq \mathscr{J}_e|_S$ on A^{**} (that is, $\mu_e \leq (\mathscr{J}_e)^{1f}$ in view of Lemma 5.1.2).

Proof. Let $\{F_\alpha\}$ be a net in the unit ball of A^{**} which converges to 0 in the \mathcal{J}_e topology (see Definition 3.2.6). Then for each f in A^* and $\epsilon > 0$ choose θ_i in P and complex numbers λ_i such that $\|f - \sum_{i=1}^4 \lambda_i \theta_i\| < \epsilon/2$.

Then there exists α_0 such that $\alpha > \alpha_0$ implies

$$\begin{aligned} & \max \{F_\alpha * F_\alpha(\theta_i)^{\frac{1}{2}}, F_\alpha F_\alpha^*(\theta_i)^{\frac{1}{2}} : i=1,2,3,4\} \\ & < \epsilon/2 \cdot (1 + \sum_{i=1}^4 |\lambda_i| \|\theta_i\|^{\frac{1}{2}}). \text{ Thus, by Lemma 3.2.4 we have} \\ & \text{for each } a \text{ in } A \text{ with } \|a\| \leq 1 \text{ and for } \alpha > \alpha_0 \text{ that} \\ & \max \{|F_\alpha \cdot f(a)|, |F_\alpha : f(a)|\} = \max\{|aF_\alpha(f)|, |F_\alpha a(f)|\} < \epsilon/2 \\ & + \max \{\sum_{i=1}^4 |\lambda_i| |aF(\theta_i)|, \sum_{i=1}^4 |\lambda_i| |F_\alpha a(f)|\} \\ & \leq \epsilon/2 + \max \{\sum_{i=1}^4 |\lambda_i| \theta_i(aa^*)^{\frac{1}{2}} \cdot F_\alpha * F_\alpha(\theta_i)^{\frac{1}{2}}, \\ & \quad \sum_{i=1}^4 |\lambda_i| F_\alpha F_\alpha^*(\theta_i)^{\frac{1}{2}} \cdot \theta_i(a^*a)^{\frac{1}{2}}\} \leq \epsilon/2 \\ & + \sum_{i=1}^4 |\lambda_i| \|\theta_i\|^{\frac{1}{2}} \cdot \max\{F_\alpha * F_\alpha(\theta_i)^{\frac{1}{2}}, F_\alpha F_\alpha^*(\theta_i)^{\frac{1}{2}}\} < \epsilon \text{ Thus,} \\ & \max \{\|F_\alpha \cdot f\|, \|F_\alpha : f\|\} < \epsilon \text{ for } \alpha > \alpha_0, \text{ and the net } \{F_\alpha\} \text{ con-} \\ & \text{verges to 0 in the } \mu_e \text{ topology (see Definition 3.1.6).} \end{aligned}$$

We remark that if A is AB^* with bounded approximate identity, then $\sigma(A^{**}, A^*)|_S \leq \mathcal{J}_e|_S$ on A^{**} in view of Theorem 3.2.14. Moreover, since the Arens multiplications are regular on A^{**} by Theorem 3.2.5, we also have that $\sigma(A^{**}, A^*) \leq \mu_e \leq \tau(A^{**}, A^*)$ by Theorem 3.1.8.

Section 2. The \mathcal{J}_e and μ_e Topologies on W^* -Algebras.

In accord with Sakai [20] we consider a W^* -algebra M

to be a B^* -algebra which is the dual of a unique Banach space F called the predual of M ; by convention F is usually identified with its natural image in M^* .

Akemann [1] proved that the strong* topology on M , generated by the collection of seminorms $\{x \mapsto \theta(x^*x + xx^*)^{\frac{1}{2}} : \theta \in F; \theta(y^*y) \geq 0, \forall y \in M\}$, and the $\tau(M, F)$ topology agree on norm balls of M by using his characterization of $\sigma(F, M)$ -compact subsets of F (see the discussion preceding Theorem 5.1.6).

If A is a Banach $*$ -algebra with bounded approximate identity, we can define a topology on A^{**} analogous to the strong* topology; however, it is easy to check that this topology is equivalent to the \mathcal{L}_e topology on A^{**} (see Definition 3.2.6). We attempted unsuccessfully to prove that if A is an AB^* -algebra, then the \mathcal{L}_e and $\tau(A^{**}, A^*)$ topologies agree on norm balls of A^{**} , but we do have a partial result.

Recall that if A is an AB^* -algebra with bounded approximate identity, then by Theorems 3.2.14 and 3.2.10 the \mathcal{L}_e topology on A^{**} is Hausdorff and $\mathcal{L}_e \leq \tau(A^{**}, A^*)$.

5.2.1 Theorem. Let A be an AB^* -algebra with bounded approximate identity, and let K be a subset of A^* . Suppose there exists θ in P with $\|\theta\| \leq 1$ such that given $\epsilon > 0$ there is $\delta > 0$ such that if F is in A^{**} with $\|F\| \leq 1$

and $\max\{F^*F(\theta), FF^*(\theta)\} < \delta$, then $|F(f)| < \epsilon$ for all f in K . Then K is relatively $\sigma(A^*, A^{**})$ -compact.

Proof. Note that K is \mathcal{J}^{lf} -equicontinuous in view of Theorem 5.1.6. We first show that K is norm-bounded. Choose $\delta > 0$ such that if F is an element of A^{**} with $\|F\| \leq 1$ and $\max\{F^*F(\theta), FF^*(\theta)\} < \delta$, then $|F(f)| < \frac{1}{2}$ for all f in K . Now, if G is an element of A^{**} with $\|G\| \leq 1$, then $\max\{(\delta/4)G^*G(\theta), (\delta/4)GG^*(\theta)\} < (\delta/4)\|G\|^2\|\theta\| < \delta$ and so $|G(f)| < \delta^{-\frac{1}{2}}$ for all f in K .

Considering K as a subset of A^{***} , we see that K is relatively $\sigma(A^{***}, A^{**})$ -compact. Let γ be a $\sigma(A^{***}, A^{**})$ -limit point of K . We now show that γ is $\sigma(A^{**}, A^*)$ -continuous on A^{**} and, therefore, is an element of A^* . This will prove that K is relatively $\sigma(A^*, A^{**})$ -compact.

To that end, let $\{H_\alpha\}$ be a net in the unit ball of A^{**} which converges to 0 in the $\tau(A^{**}, A^*)$ topology and let $\epsilon > 0$. By hypothesis choose $\delta > 0$ associated to this ϵ . Since the net $\{H_\alpha\}$ converges to 0 in the \mathcal{J}_e topology by Theorem 3.2.10, there exists α_0 such that $\max\{H_\alpha^*H_\alpha(\theta), H_\alpha H_\alpha^*(\theta)\} < \delta$ whenever $\alpha > \alpha_0$. Consequently, $|\gamma(H_\alpha)| \leq \sup\{H_\alpha(f) : f \in K\} \leq \epsilon$ for $\alpha > \alpha_0$. Therefore, $\ker \gamma \cap S$ is $\tau(A^{**}, A^*)$ -closed, where S denotes the unit ball of A^{**} ; and it follows that $\ker \gamma \cap S$ is $\sigma(A^{**}, A^*)$ -closed.

Since A^* is a Banach space, γ is $\sigma(A^{**}, A^*)$ -continuous by [18, Theorem 5, p.111; Theorem 1, p. 26].

5.2.2 Corollary. If A is a B^* -algebra, then a subset K of A^* is relatively $\sigma(A^*, A^{**})$ -compact if and only if K satisfies the condition stated in the theorem. Consequently,

$$\rho_e|_S = \tau(A^{**}, A^*)|_S, \text{ where } S \text{ denotes the unit ball of } A^{**}.$$

Proof. Since the bidual A^{**} of A is a W^* -algebra under the Arens multiplication $(F, G) \rightarrow FG$ in view of Theorem 3.2.16, and since a relatively $\sigma(A^*, A^{**})$ -compact set is norm-bounded [18, p.67], the corollary follows from [1] and the observation that the $\tau(A^{**}, A^*)$ topology is that of uniform convergence on absolutely convex $\sigma(A^*, A^{**})$ -compact sets.

A paper by P. C. Shields [24] was recently brought to our attention in which he introduced the topology on a W^* -algebra M generated by the seminorms $x \mapsto \max\{\|f \cdot x\|, \|x \cdot 1\|\}$ (in our notation), where f is in the predual of M : this topology is the W^* -analogue of our μ_e topology on the bidual of a Banach algebra with bounded approximate identity (see Definition 3.1.6), and for that reason we will refer to it as the μ_e topology.

What interests us is that Shields claims the mapping $(a, b) \mapsto ab$ is μ_e -continuous on a W^* -algebra for $\|a\| \leq 1$. This is not necessarily true in the case of a non-commutative

W^* -algebra as Example 5.2.6 will show. We also include a proof that if a B^* -algebra A satisfies the property that the mapping $(a,b) \rightarrow ab$ is μ -continuous for $\|a\| \leq 1$, then $(A,\mu)^\wedge$ equals A^{**} and so is an algebra under either Arens multiplication (see the remarks following Lemma 4.3.6). This result also generalizes Theorem 4.3.9: See Corollary 5.2.10.

5.2.4 Theorem. Let M be a W^* -algebra with predual F . Suppose there exists an element θ in F such that $\theta(x^*x) \geq 0$ for all x in M and $\epsilon > 0$ which satisfy the following property: for every φ in F such that $\varphi(x^*x) \geq 0$ for all x in M and for $\delta > 0$ there exist equivalent projections p and q in M (that is, $p^2 = p = p^*$, $q^2 = q = q^*$, and there exists an element v in M for which $p = vv^*$ and $q = v^*v$) so that $\theta(p) > \epsilon$ and $\varphi(q) < \delta$. Then the mapping $(a,b) \rightarrow ab$ on M is not μ_ϵ -continuous for $\|a\| \leq 1$.

Proof. Let U be the μ_ϵ -neighborhood $\{x \in M : \|\theta \cdot x\| \leq \epsilon, \|x \cdot \theta\| \leq \epsilon\}$, and let $V = \{x \in M : \|\psi_k \cdot x\| \leq \epsilon_1, \|x \cdot \psi_k\| \leq \epsilon_1, k = 1, \dots, K\}$ and $W = \{x \in M : \|\varphi_n \cdot x\| \leq \epsilon_2, \|x \cdot \varphi_n\| \leq \epsilon_2, n = 1, \dots, N\}$ be any two basis μ_ϵ -neighborhoods. Choose $\lambda > 0$ such that $\lambda \leq \max\{1, \epsilon_1 / (\|\psi_1\| + \dots + \|\psi_K\|)\}$. Let φ equal $\sum_{n=1}^N \varphi_n$ and $\delta^{\frac{1}{2}}$ equal $\lambda \epsilon_2 / (\|\varphi_1\|^{\frac{1}{2}} + \dots + \|\varphi_N\|^{\frac{1}{2}})$. Then by hypothesis for φ and δ there exist projections p and q in M and an element v in M such that

(i) $vv^* = p$ and $v^*v = q$; and (ii) $\theta(p) > \epsilon$ and $\varphi_n(q) \leq \varphi(q) < \delta$ for $n = 1, \dots, N$.

Since $\max\{\|\lambda v \cdot \psi_k\|, \|\psi_k \cdot \lambda v\|\} \leq \lambda \|\psi_k\| \|v\| \leq \lambda \|\psi_k\| \leq \epsilon_1$ for $k = 1, \dots, K$, and $\lambda \leq 1$, then λv is an element of $V \cap S$, where S denotes the unit ball of M . Furthermore, as a consequence of the Cauchy-Schwartz inequality we have that $\max\{\|(1/\lambda)q \cdot \varphi_n\|, \|\varphi_n \cdot (1/\lambda)q\|\} \leq (1/\lambda) \|\varphi_n\|^{\frac{1}{2}} \varphi_n(q)^{\frac{1}{2}} \leq (1/\lambda) \|\varphi_n\|^{\frac{1}{2}} \leq \epsilon_2$, for $n = 1, \dots, N$; hence $(1/\lambda)q$ is an element of W . Therefore, $vq = \lambda v \cdot (1/\lambda)q$ is an element of $(V \cap S) \cdot W$.

Noting that $v^*vv^* = v^*$ by [3, p.153], we see that $\|\theta \cdot vq\| \geq \theta(vq^*v^* = \theta(vv^*vv^*) = \theta(vv^*) = \theta(p) > \epsilon$. Thus, vq is not an element of U . Since V and W are arbitrary basic μ_e -neighborhoods, the mapping $(a, b) \rightarrow ab$ on M cannot be μ_e -continuous for $\|a\| \leq 1$.

5.2.5 Corollary. If M is a W^* -algebra containing an infinite number of non-zero equivalent orthogonal projections, then the mapping $(a, b) \rightarrow ab$ on M is not μ_e -continuous for $\|a\| \leq 1$.

Proof. Let $\{p_n\}$ be an infinite set of non-zero equivalent orthogonal projections. There exists θ in F such that $\theta(x^*x) \geq 0$ for all x in M and $\theta(p_1) > 0$: in fact, F is spanned by the set $\{\varphi \in F : \varphi(x^*x) \geq 0, \forall x \in M\}$ [20].

Choose $\epsilon > 0$ such that $\epsilon < \theta(p_1)$. Let φ be any element of F with the property that $\varphi(x^*x) \geq 0$ for all x in M and let $\delta > 0$. Then there exists a positive integer n such that $\varphi(p_N) < \delta$. If this were not true, choose a positive integer n such that $n\delta \geq \|\varphi\|$; then $p_1 + \dots + p_n$ is a projection and so has norm 1, but by hypothesis $n\delta \leq \varphi(p_1 + \dots + p_n) \leq \|\varphi\|$, a contradiction. The corollary now follows from the theorem.

5.2.6 Example. Let H be an infinite dimensional Hilbert space. Then $B(H)$ is a W^* -algebra (in fact, it is the bidual of a B^* -algebra; namely, the compact operators on H). However, the mapping $(a,b) \rightarrow ab$ on $B(H)$ is not μ_e -continuous for $\|a\| \leq 1$. To see this, let $\{x_n : n=1,2,\dots\}$ be an orthonormal set of vectors in H , and let p_n be the projection on the closed subspace of H generated by x_n . Clearly, the sequence $\{p_n\}$ satisfies the hypothesis of Corollary 5.2.5, and the desired conclusion now follows.

We remark that the mapping $(a,b) \rightarrow ab$ for $\|a\| \leq 1$ on a type III factor also fails to be μ_e -continuous in view of Corollary 5.2.5.

In conclusion we will prove that if A is a B^* -algebra such that the mapping $(a,b) \rightarrow ab$ on A is μ -continuous for $\|a\| \leq 1$, then $(A,\mu)^\wedge$ equals A^{**} and, hence, is an algebra under either Arens multiplication. The proof is quite

complicated and in part is based on the fact that A^{**} is a W^* -algebra under the Arens multiplication $(F,G) \rightarrow FG$ (see Theorem 3.2.16). To facilitate the proof, we assume that the reader is familiar with Sakai's result that a W^* -algebra can be considered as a von Neumann algebra [20]. Therefore, we use certain terminology freely and cite references from [6].

5.2.7 Lemma. Let A be a B^* -algebra and let $P_0 = \{\theta \in A^*: \|\theta\| \leq 1; \theta(x^*x) \geq 0, \forall x \in A\}$. Suppose that $\{F_\alpha\}$ is a net of hermetian elements in the unit ball of A^{**} such that $\lim F_\alpha F_\alpha(\theta) = 0$ for some θ in P_0 . If Q denotes the support of θ in A^{**} , then the net $\{QF_\alpha Q\}$ converges to 0 in the \mathcal{L}_e topology.

Proof. In light of our earlier remarks it suffices to prove the lemma for a von Neumann algebra M . That is, we must prove the following: if $\{F_\alpha\}$ is a net of hermetian elements in the unit ball of M such that $\lim F_\alpha F_\alpha(\theta) = 0$ for some normal positive linear functional θ in M , then the net $\{QF_\alpha Q\}$ converges to 0 in the \mathcal{L}_e topology, where Q denotes the support of θ . For the definition of support of a normal positive linear functional on M see [6, p.57]. Note also that the \mathcal{L}_e topology is generated by the seminorms $F \mapsto \max\{F^*F(\varphi), FF^*(\varphi)\}$, where φ is a normal positive linear functional.

By Proposition 4 of [6 ,p.58] the net $QF_\alpha Q$ converges to 0 in the strong operator topology: that is, if H denotes the Hilbert space on which M is a ring of operators, then $\lim \|QF_\alpha Qx\| = 0$ for every x in H . If φ is any normal positive linear functional on M $\varphi = \sum_{i=1}^{\infty} [x_i, x_i]$, where $\{x_i\}$ is a subset of H such that $\sum_{i=1}^{\infty} \|x_i\|^2 < \infty$ and $[x_i, x_i]$ is a linear functional on M defined by $[x_i, x_i](F) = ((Fx_i, x_i))$.

For $\epsilon > 0$ choose a positive integer N such that $\sum_{i=N+1}^{\infty} \|x_i\|^2 < \epsilon/2$. Since the net $\{QF_\alpha Q\}$ converges to 0 in the strong operator topology, there exists α_0 such that $\alpha > \alpha_0$ implies that $\|QF_\alpha Qx_i\| < \epsilon/2N$ for $i = 1, \dots, N$. By noting that $QF_\alpha Q$ is hermetian and has norm at most 1 for each α , we have for $\alpha > \alpha_0$ that $QF_\alpha Q QF_\alpha Q(\varphi) = \sum_{i=1}^N \|QF_\alpha Qx_i\|^2 + \sum_{i=N+1}^{\infty} \|QF_\alpha Qx_i\|^2 \leq N(\epsilon/2N) + \sum_{i=N+1}^{\infty} \|QF_\alpha Q\|^2 \|x_i\|^2 < \epsilon$. Therefore, we see that the net $\{QF_\alpha Q\}$ converges to 0 in the \mathscr{S}_e topology.

5.2.8 Lemma. Let A be a B^* -algebra such that the mapping $(a, b) \rightarrow ab$ is μ -continuous for $\|a\| \leq 1$, and let S' denote the unit ball of A^{**} . Then for each μ_e -neighborhood U' in A^{**} there exists a μ -neighborhood V in A and a μ_e -neighborhood W' in A^{**} such that the set $V \cdot (W' \cap S') = \{aF : a \in V, F \in W' \cap S'\}$ is contained in U' .

Proof. Since the Arens multiplications are regular on

A by Theorem 3.2.5, we have that A^{**} is contained in $(A, \mu)^\wedge$ and that the μ -completion topology restricted to A^{**} is equivalent to the μ_e topology on A^{**} by Theorem 4.3.7 and Corollary 4.3.8. Thus, there exists a μ -neighborhood U in A such that $U^{\circ\circ}$ (taken in A^{**}) is contained in $\frac{1}{2}U'$.

On A the mapping $(a, b) \rightarrow ab$ on A for $\|b\| < 1$ is also μ -continuous. If S denotes the unit ball of A , we can choose μ -neighborhoods V and W such that the set $V \cdot (W \cap S) = \{ab : a \in V, b \in W \cap S\}$ is contained in U . It is clear that $U^\circ \cdot V$ is contained in $(W \cap S)^\circ$ and that $V \cdot (W \cap S)^{\circ\circ}$ is contained in $U^{\circ\circ}$, where the bipolars are taken in A^{**} .

Let $W' = W^{\circ\circ}$ (in A^{**}); then W' is a μ_e -neighborhood by Corollary 4.3.8. Note that $(W \cap S)^\circ$ in A^* is the $\sigma(A^*, A)$ -closed absolutely convex hull of $W^\circ U S^\circ$ [18, p.36]; and since W° and S° are $\sigma(A^*, A^*)$ -compact [18, p.61], we have that $(W \cap S)^\circ$ is contained in $W^\circ + S^\circ$. It is easy to check that $W' \cap S'$ is contained in $2(W \cap S)^{\circ\circ}$ in A^{**} . Therefore, $V(W' \cap S') \subset V 2(W \cap S)^{\circ\circ} \subset 2U^{\circ\circ} \subset 2U^{\circ\circ} \subset U'$.

5.2.9 Lemma. If A is a B^* -algebra, and if a net $\{a_\alpha\}$ in A converges to T in $(A, \mu)^\wedge$ in the completion topology, then for each f in A^* we have that $|T(f)| \leq \|T \cdot f\| = \lim \|a_\alpha \cdot f\|$.

Proof. Considering T as an element of $(A, \mu^2)^\wedge$ by Lemma 4.3.6, we note that $T \cdot g$ is an element of A^* for every

g in A^* by Lemma 4.3.2. Let $\epsilon > 0$ and let $U = \{x \in A : \|x \cdot f\| \leq \epsilon\}$ be a μ^2 -neighborhood in A . Then $U^{oo} \cap (A, \mu)^\wedge$ is a μ -completion neighborhood in view of Corollary 4.1.3, where U^{oo} is taken in A^{**} . Moreover, $U^o = \{F : f : F \in A^{**}, \|F\| \leq 1/\epsilon\}$ by Lemma 4.3.1. Therefore, there exists α_0 such that $\alpha > \alpha_0$ implies $T - a_\alpha$ is an element of $U^{oo} \cap (A, \mu)^\wedge$.

In particular, for each x in A with $\|x\| \leq 1$, we have that $|(T - a_\alpha)(f \cdot x)| = |(T - a_\alpha)(1/\epsilon)x : f| \leq \epsilon$ whenever $\alpha > \alpha_0$. Hence, $\|(T - a_\alpha) \cdot f\| \leq \epsilon$ for $\alpha > \alpha_0$, and $\|T \cdot f\| = \lim \|a_\alpha \cdot f\|$.

Finally, by the remarks following Theorem 3.2.14, there exists an element I in A^{**} which is an identity for either Arens multiplication. Moreover, $\|I\| = 1$ by Lemma 3.1.5, and $I : g = g$ for every g in A^* since $F(I : g) = F : I(g) = F(g)$ for every F in A^{**} . Considering T as an element of $(A, \mu^2)^\wedge$ in view of Lemma 4.3.6, we have by Lemma 4.3.2 that $|T(f)| = |T(I : f)| = |I(T \cdot f)| \leq \|I\| \|T \cdot f\| \leq \|T \cdot f\|$.

5.2.10 Theorem. If A is a B^* -algebra such that the mapping $(a, b) \rightarrow ab$ is μ -continuous for $\|a\| \leq 1$, then $(A, \mu)^\wedge$ equals A^{**} .

Proof. From Theorems 3.2.5 and 4.3.7 we have that A^{**} is a subset of $(A, \mu)^\wedge$. To show the other containment,

it suffices to show that if T is an element of $(A, \mu)^\wedge$ and if $\{\theta_n\}$ is a sequence in A^* such that $\theta_n(x^*x) \geq 0$ for all x in A and $\|\theta_n\| \leq 1/2^n$, then $\lim T(\theta_n) = 0$: to see this, note that each f in A^* can be written as $f = \varphi_1 - \varphi_2 + i\varphi_3 - i\varphi_4$, where $\varphi_1(x^*x) \geq 0$ for all x in A^* and $\|\varphi_1\| + \|\varphi_2\| + \|\varphi_3\| + \|\varphi_4\| \leq 2\|f\|$ by [7 ,p.40].

Let θ equal $\sum_{n=1}^{\infty} \theta_n$; then θ is a well-defined element of A^* . Moreover, $0 \leq F^*F(\theta_n) \leq F^*F(\theta)$ for each F in A^{**} and each integer n by Lemma 3.2.4. Considering A^{**} as a W^* -algebra (under the Arens multiplication $(F, G) \rightarrow FG$), we can choose a sequence $\{G_n\}$ in A^{**} such that $0 \leq G_n \leq 1$ and $F(\theta_n) = G_n F G_n(\theta)$ for all F in A^{**} by [6 ,p.63]. If Q denotes the support of θ , then $F(\theta) = Q F Q(\theta)$ and $Q F Q(\theta_n) = F(\theta_n)$ for each F in A^{**} and each n [6 ,pp.58-9]. Note also that $\lim Q G_n Q Q G_n Q(\theta) = \lim G_n Q G_n(\theta) = \lim Q(\theta_n) \leq \lim \|\theta_n\| = 0$ since Q is the support of θ . Consequently, the sequence $\{Q G_n Q\}$ of hermetian elements in the unit ball of A^{**} converges to 0 in the \mathscr{A}_e topology by Lemma 5.2.7. Moreover, by Theorem 5.1.10 we have that $\{Q G_n Q\}$ converges to 0 in the μ_e topology.

Now let $\epsilon > 0$ and consider the μ_e -neighborhood $U' = \{F \in A^{**} : \|F \cdot \theta\| \leq \epsilon/2, \|F : \theta\| \leq \epsilon/2\}$ in A^{**} . By Lemma 5.2.8 there exists a μ -neighborhood V in A and a μ_e -neighborhood W' in A^{**} such that the set $\{aF \in A^{**} : a \in V; F \in W', \|F\| \leq 1\}$ is contained in U' . Moreover,

V can be chosen to be $\sigma(A, A^*)$ -closed and absolutely convex [18, p.12].

Let T be some element of $(A, \mu)^\wedge$, and choose a net $\{a_\alpha\}$ in A converging to T in the completion topology. Since $\frac{1}{2}V^{\circ\circ} \cap (A, \mu)^\wedge$ is a completion neighborhood, where $V^{\circ\circ}$ is taken in $A^{*'}$, there exists α_0 such that $\alpha \geq \alpha_0$ implies $T - a_\alpha$ is an element of $\frac{1}{2}V^{\circ\circ} \cap (A, \mu)^\wedge$. Considering A as a subset of A^{**} and letting $a_0 = a_{\alpha_0}$, we then have that $a_\alpha - a_0$ is an element of $V^{\circ\circ} \cap A$ for $\alpha > \alpha_0$. Since V is absolutely convex and $\sigma(A, A^*)$ -closed, it is easy to see that $V^{\circ\circ} \cap A$ is equal to V .

Since $\lim \|\theta_n\| = 0$, there exists a positive integer N_0 such that $\|\theta_n\| < \epsilon/2 \cdot (\|a_0\| + 1)$ for $n > N_0$. Furthermore, since W' is a μ_e -neighborhood and the sequence $\{QG_nQ\}$ converges to 0 in the μ_e -topology, there exists a positive integer N_1 such that QG_nQ is in W' whenever $n > N_1$. Let $N_0 = N + N_1$.

Consequently, for $\alpha > \alpha_0$ and $n > N_0$ we have that $a - a_0$ is an element of V and QG_nQ is an element of W' such that $\|QG_nQ\| \leq 1$; therefore, $(a_\alpha - a_0)QG_nQ$ is an element of $U' = \{F \in A^{**} : \|F \cdot \theta\| \leq \epsilon/2, \|F : \theta\| \leq \epsilon/2\}$. If $F_{\alpha,n}$ denotes $(a_\alpha - a_0)QG_nQ$ for $\alpha > \alpha_0$ and $n > N_0$, then $\|QG_nQ : (F_{\alpha,n} \cdot \theta)\| \leq \|QG_nQ\| \|F_{\alpha,n} \cdot \theta\| \leq \epsilon/2$.

Recalling the relations on Q and G_n , we have for x in

A that $QG_n Q: (F_{\alpha, n} \cdot \theta)(x) = QG_n Q(x \cdot (F_{\alpha, n} \theta)) = QG_n Q(x F_{\alpha, n} \cdot \theta)$
 $= QG_n Qx(a_\alpha - a_0)QG_n Q(\theta) = G_n Qx(a_\alpha - a_0)QG_n(\theta) = Qx(a_\alpha - a_0)Q(\theta_n)$
 $= \theta_n(x(a_\alpha - a_0)) = (a_\alpha - a_0) \cdot \theta_n(x)$. Hence, $\|(a_\alpha - a_0) \cdot \theta_n\|$
 $= QG_n Q: (F_{\alpha, n} \cdot \theta)\| \leq \epsilon/2$ for $\alpha > \alpha_0$ and $n > N_0$. Thus,
 $\|a_\alpha \cdot \theta_n\| \leq \epsilon/2 + \|a_0 \cdot \theta_n\| \leq \epsilon/2 + \|a_0\| \|\theta_n\| < \epsilon/2 + \epsilon/2$ whenever
 $\alpha > \alpha_0$ and $n > N_0$. By Lemma 5.2.9 we now have that
 $|T(\theta_n)| \leq \|T \cdot \theta_n\| = \lim \|a_\alpha \cdot \theta_n\| = 0$ for $n > N_0$. The proof is
 thereby completed.

5.2.11 Corollary. If A is a commutative B^* -algebra, then
 $(A, \mu)^\wedge$ equals A^{**} .

Proof. By the theorem it suffices to show that the
 mapping $(a, b) \rightarrow ab$ is μ -continuous for $\|a\| \leq 1$. Let $\{a_\alpha\}$
 and $\{b_\alpha\}$ be nets in A converging to a and b , respectively,
 where $\|a_\alpha\| \leq 1$. Then for each f in A^* we have that
 $\lim \|a_\alpha b_\alpha \cdot f - ab \cdot f\| \leq \lim \|a_\alpha b_\alpha \cdot f - a_\alpha b \cdot f\| + \lim \|a_\alpha b \cdot f - ab \cdot f\|$
 $\leq \lim \|b_\alpha \cdot f - b \cdot f\| + \lim \|(a_\alpha - a) \cdot (b \cdot f)\| = 0$ and
 $\lim \|f \cdot a_\alpha b_\alpha - f \cdot ab\| = \lim \|a_\alpha b_\alpha \cdot f - ab \cdot f\| = 0$ from the
 commutativity of multiplication on A .

CHAPTER VI

QUESTIONS AND PROBLEMS

Several interesting questions and problems have arisen in connection with this paper. We list them here and reference those sections of the preceding chapters which are applicable.

6.1.1 Problem. Characterize those Banach algebras for which the Arens multiplications are regular.

Theorem 3.1.8 as well as [2], [10], and [11] give several necessary and sufficient conditions for the Arens multiplications to be regular on the bidual of a Banach algebra. With regard to this problem Civin and Yood [4] have studied algebraic properties of the biduals of various Banach algebras and have given several interesting examples. Hennefeld [] has looked at this problem in Banach spaces which are separable and uniformly convex.

6.1.2 Question. If A is a Banach algebra and if there exists an identity I in A^{**} for either Arens multiplication, is the norm of I equal to 1?

Recall that A^{**} under either Arens multiplication is a Banach algebra; but if A^{**} has an identity, we cannot

prove that it has norm 1. If A has a bounded approximate identity, then the question is answered in the affirmative by Lemma 3.1.5.

6.1.3 Question. If A is a Banach algebra with bounded approximate identity, is the (non-empty) set of right identities in A^{**} for either Arens multiplication a subset of the unit ball of A^{**} ? Moreover, does a right identity have to be a $\sigma(A^{**}, A^*)$ -cluster point of the bounded approximate identity?

Lemmas 3.1.4 and 3.1.5 and the discussion following these lemmas are related to this question.

6.1.4 Question. If \mathfrak{J} is a locally convex Hausdorff topology on a Banach algebra A such that $(A, \mathfrak{J})^*$ is a subset of A^* , what conditions on \mathfrak{J} and A imply that either of the Arens multiplications can be defined on $(A, \mathfrak{J})^\wedge$ and that $(A, \mathfrak{J})^\wedge$ is an algebra under this multiplication? Moreover, what additional properties on \mathfrak{J} ensure that $(A, \mathfrak{J})^\wedge$ is an algebra under both Arens multiplications and that these multiplications are regular?

6.1.5 Question. If \mathfrak{J} and \mathfrak{J}' are two locally convex topologies on a Banach algebra A such that $(A, \mathfrak{J})^*$ and $(A, \mathfrak{J}')^*$ are contained in A^* and such that $(A, \mathfrak{J})^\wedge$ and $(A, \mathfrak{J}')^\wedge$ are algebras under some Arens multiplication, is

$(A, \mathfrak{A}\mathfrak{V}\mathfrak{A}')^\wedge$ an algebra under some Arens multiplication?

Theorem 4.1.5 gives a partial answer to Question 6.1.4. In Chapter IV we examine both Questions 6.1.4 and 6.1.5 from the standpoint of examples. Requiring only that A have a bounded approximate identity, we show that $(A, \beta^i)^\wedge$ and $(A, \mu^i)^\wedge$, for $i=1,2$, are algebras under some Arens multiplication. If, in addition, the positive linear functionals in A separate points of A (or, equivalently, if the \mathfrak{A} topologies are Hausdorff), then $(A, \mathfrak{A})^\wedge = (A, \mathfrak{A}^i)^\wedge (i=1,2)$ is an algebra under both Arens multiplications and these multiplications are regular. We need additional assumptions (see Theorems 4.2.14, 4.3.9, and 5.2.10) to prove that $(A, \beta)^\wedge$ and $(A, \mu)^\wedge$ are algebras under either Arens multiplication. In particular, for μ we impose severe restrictions on A : namely, that A be a B^* -algebra on which the multiplication $(a,b) \rightarrow ab$ is μ -continuous for $\|a\| \leq 1$. In this connection we ask the following:

6.1.6 Question. How restrictive is the condition on a Banach algebra A that the multiplication $(a,b) \rightarrow ab$ be μ -continuous for $\|a\| \leq 1$? A less general, but analogous, question is to ask how restrictive is the condition on a W^* -algebra M that the multiplication $(a,b) \rightarrow ab$ be μ_e -continuous for $\|a\| \leq 1$.

In either case if the algebra is commutative, the

condition imposes no restriction: the proofs are similar to that of Corollary 5.2.11. Partial results for the non-commutative W^* -algebra case are given in Theorem 5.2.4 and Corollary 5.2.5.

6.1.7 Question. Is there a single technique to show that the completions of A with respect to β^i , μ^i , and ν^i , for $i = 1, 2$, are algebras under some Arens multiplication? Are there more general conditions than imposed by Theorems 4.2.14 and 5.2.10 which imply that $(A, \beta)^\wedge$ and $(A, \mu)^\wedge$ are algebras under either Arens multiplication?

The discussion following Theorem 4.4.8 is germane to this question.

6.1.8 Problem. Investigate the properties of AB^* -algebras.

In section 3 of Chapter III we examine some of the properties of AB^* -algebras; one of the more interesting is that a c_0 -direct sum of AB^* -algebras is AB^* , while a c_0 -direct sum of algebras equivalent to B^* -algebras can fail to be equivalent to a B^* -algebra (see Examples 3.3.7 and 3.3.8). The study of AB^* -algebras might be useful in this context, and other applications would be worthwhile. Two important properties of AB^* -algebras to establish is whether they have a bounded approximate identity and whether they are symmetric.

6.1.9 Question. Is there an intrinsic proof that the bidual of a B^* -algebra A is also a B^* -algebra (that is, a proof which does not require that A or A^{**} be represented as a set of operators on some Hilbert space)?

Tomita's proof [27] of this result (see Theorem 3.2.16) fails to be intrinsic at only one point, the relation $FG(\theta) = \overline{G^*F^*(\theta)}$ for all F and G in A^{**} and for all positive linear functionals θ on A is needed to prove that the mapping $F \rightarrow F^*$ on A^{**} is an involution; and this relation is established by using Theorem 3.2.2, which asserts that every positive linear functional on A generates a $*$ -representation of A^{**} . This relation for F and G when F and G are elements of A is an easy consequence of the fact that each θ is positive on A .

6.1.10 Question. What conditions on a Banach algebra A are necessary and sufficient that A be a two-sided ideal in A^{**} under either Arens multiplication (that is, $A^{**} = \mathcal{J}(A)$)?

We prove in Theorem 4.2.17 that if A is a B^* -algebra, then $A^{**} = \mathcal{J}(A)$ if and only if A is dual. This question can be phrased in certain Banach spaces, and Hennefeld [12] has studied this question.

6.1.11 Question. If \mathfrak{T} is a locally convex topology on a Banach algebra A such that $(A, \mathfrak{T})^*$ is contained in A^* , what

conditions on A and \mathfrak{J} imply that \mathfrak{J} -equicontinuous subsets in A^* are relatively $\sigma(A^*, A^{**})$ -compact? Conversely, what conditions on A and \mathfrak{J} imply that relatively $\sigma(A^*, A^{**})$ -compact subsets of A^* are \mathfrak{J} -equicontinuous?

We know that if A has a bounded approximate identity and if the positive linear functionals on A separate points of A , then \mathfrak{J}^1 , \mathfrak{J}^2 , \mathfrak{J} , and \mathfrak{J}^{lf} - equicontinuous sets are always relatively $\sigma(A^*, A^{**})$ -compact in view of Corollary 3.2.9 and Theorem 5.1.6. By our generalization of D. C. Taylor's characterization of β -equicontinuous sets (see the proof of Theorem 5.1.3 and [26]) we see that if A is a Banach algebra with bounded approximate identity such that $(A, \beta)^* = A^*$ and $A^{**} = \mathfrak{J}(A)$, then relatively $\sigma(A^*, A^{**})$ -compact subsets of A^* are β -equicontinuous. If A is a B^* -algebra, then a subset of A^* is relatively $\sigma(A^*, A^{**})$ -compact if and only if it is \mathfrak{J}^{lf} -equicontinuous by Corollary 5.2.2. Furthermore, μ -equicontinuous subsets of A^* are relatively $\sigma(A^*, A^{**})$ -compact if and only if the Arens multiplications on A^{**} are regular by Theorem 3.1.8 and Lemma 4.3.1.

6.1.12 Question. If \mathfrak{J} is a locally convex Hausdorff topology on a Banach algebra A such that $\mathfrak{J} \leq \tau(A, A^*)$ is it true that $(A, \mathfrak{J})^*$ in A^* is norm-dense in $(A, \mathfrak{J}^{lf})^*$ and that $(A, \mathfrak{J}^{lf})^*$ is norm-closed? What conditions on A and \mathfrak{J}

imply that $\mathfrak{T} = \mathfrak{T}^{lf}$?

For the topologies β^1 , β^2 , β , and \mathscr{J} the first question is answered in the affirmative by Theorems 2.1.7, 5.1.4, Corollary 5.1.7, and the remarks preceding Theorem 5.1.3. Moreover, the facts that $\beta = \beta^{lf}$ and that $\beta^i = (\beta^i)^{lf}$, for $i=1,2$, are established. If A is a B^* -algebra, then $(A, \mathscr{J})^* = (A, \mathscr{J}^{lf})^*$ in view of Corollary 5.1.7 and in contrast to Example 5.1.8. Perhaps in this case $\mathscr{J} = \mathscr{J}^{lf}$: then we would also have that $\mathscr{J}_e = \tau(A^{**}, A^*)$ on A^{**} in view of Corollaries 5.2.2 and 3.2.9.

6.1.3 Question. Can the μ_e and \mathscr{J}_e topologies on W^* -algebras be used in the non-spatial approach to the study of W^* -algebras (that is, the study of W^* -algebras as B^* -algebras which are the duals of Banach spaces and not as weakly-closed $*$ -subalgebras of operators on some Hilbert space)?

We see in the proof of Lemma 5.2.7 that certain norm-bounded SOT-convergent nets also converge in the \mathscr{J}_e and, hence, the μ_e topologies by Theorem 5.1.10. Furthermore, P. C. Shields [23] showed on $B(H)$ that $\mu_e \leq \text{SOT}$. We believe that the μ_e and \mathscr{J}_e topologies are non-spatial analogues for the ultrastrong and ultraweak (operator) topologies. Much work remains to be done in this area.

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